Powers of a Matrix of Special Type

By Gilbert C. Best

In this paper it is shown that if $ETE = I$, i.e., if the columns of $E$ are orthonormal, and $K$ is a diagonal matrix with all terms positive, then

$$[I + E(K - I)E^T]^n = I + E(K^n - I)E^T$$

for any real $n$. Since given any matrix $V$ and a positive definite matrix $G$ it is possible to find an $E$ and $K$ as just described satisfying

$$VGV^T = E(K - I)E^T;$$

this provides a method for finding any power or root of matrices of the type

$$B = I + VGV^T.$$

This becomes particularly useful for work on high speed digital computers when $G$ is very small. For suppose $V$—and hence also $E$—is $n \times r$ with $n \gg r$ and $G$ is $r \times r$. Then keeping only $E$ and, trivially, the diagonal of $K$, in fast-access storage and using only $r$ core locations as working storage one can perform a rapid multiplication of an $n \times 1$ vector by any of the matrices $B, B^{1/2}, B^{-1/2}, B^{-1}$, etc., with $B$ as in (3).

Equation (1) can be easily proved from the identity*

$$[I + E(K^p - I)E^T][I + E(K^q - I)E^T] = I + E(K^{p+q} - I)E^T$$

which can be established by merely multiplying out the terms on the left side and making use of the relation $E^TE = I$.

The conversion indicated in Eq. (2) can be accomplished by orthogonalizing the columns of $V$ by elementary column operations [1]—the round-off error problem [2], not being significant when $r$ is small—to obtain the matrix $O$ such that

$$OT^O = I.$$ 

Let the product of the corresponding elementary matrices be the matrix $R$, i.e.,

$$O = VR$$

or

$$V = OR^{-1}.$$ 

Then let

$$H = R^{-1}GR^{-1T}.$$ 

Note that $H$ is $r \times r$ and symmetric positive definite. One then solves the small eigenproblem

$$HX = XL$$

for $X$ and the diagonal matrix $L$. It follows that

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* The author is indebted to the referee for this equation.
Then, using Eqs. (7) through (11)
\begin{equation}
V G V^T = O H O^T = O X L X^T O^T = E L E^T
\end{equation}
where
\begin{equation}
E = O X .
\end{equation}
Furthermore \(E^T E = X^T O^T O X = X^T X = I\) using Eqs. (13), (5), and (10).

An alternative method, though somewhat longer but more stable in terms of rounding errors, is to perform a Cholesky factorization \([3]\) of the positive definite matrix \(G\), obtaining
\begin{equation}
G = U U^T .
\end{equation}
Then form the matrix
\begin{equation}
C = U^T V V U
\end{equation}
and solve the small \(r \times r\) eigenproblem
\begin{equation}
C Y = Y L
\end{equation}
for \(Y\) and \(L\). Since \(C\) is positive definite all terms of \(L\) are positive and one may set
\begin{equation}
E = V U Y L^{-1/2}
\end{equation}
so that
\begin{equation}
E^T E = L^{-1/2} Y^T U U^T Y U Y L^{-1/2} = L^{-1/2} Y^T C Y L^{-1/2} = L^{-1/2} L L^{-1/2} = I
\end{equation}
and
\begin{equation}
E L E^T = V U Y L^{-1/2} L L^{-1/2} Y^T U U^T = V G V^T
\end{equation}
as desired. Then let
\begin{equation}
K = I + L
\end{equation}
for \(K\) as in Eq. (2).

Another identity related to (4), though not so general, for \(M\) and \(K\) diagonal matrices of nonzero terms and \(F\) such that \(F^T M F = I\), is
\begin{equation}
[M + M F(K - I) F^T M][M^{-1} + F(K^{-1} - I) F] = I
\end{equation}
which again can be proved by simply multiplying out the terms of the product.