On a Generalization of the Midpoint Rule*

By Franz Stetter

I. Introduction. A modified midpoint rule for the approximate calculation of weighted integrals \( \int_a^b p(x)f(x)dx \), where \( p(x) \geq 0 \) is the weight function, has been recently proposed by Jagermann [1]. Although this formula reduces to the common midpoint rule in the particular case \( p(x) = 1 \), in the general case of arbitrary weight functions the error does not vanish for all polynomials \( \alpha + \beta x \). The purpose of this paper is to generalize the midpoint rule such that the formula is exact for polynomials of first degree and arbitrary weight function \( p(x) \geq 0 \).

In view of practical calculations, the repeated midpoint rule is very useful because of its simplicity and small round-off error. Moreover, an error estimation does not require higher derivatives whose bounds are often not easy to obtain. For a comparison of the repeated midpoint rule to both Gaussian quadratures and “best” quadratures we refer to Stroud and Secrest [2].

II. Generalized Midpoint Rule. We assume that the weight function \( p(x) \) does not identically vanish on any subinterval of \( [a, b] \). Let

\[
y = H(x) = \int_a^x p(t)dt, \quad H(b) = 1,
\]

and let the inverse function of \( H \) (which exists because \( H(x) \) is monotonic increasing) be denoted by \( L \):

\[
x = L(y) = H^{-1}(y).
\]

For \( i = 0, 1, \ldots, N - 1 \), \( (N \geq 1) \) we put

\[
a_i = N \int_{i/N}^{(i+1)/N} L(y)dy = N \int_{x_i}^{x_{i+1}} tp(t)dt,
\]

where \( x_i = L(i/N) \). We now define the generalized rule by:

\[
\int_a^b p(x)f(x)dx = \frac{1}{N} \sum_{i=0}^{N-1} f(a_i) + R_N.
\]

Assuming \( f \in C^2[a, b] \) the error \( R_N \) can be expressed by

\[
R_N = \frac{1}{2} \left( \int_a^b x^2p(x)dx - \frac{1}{N} \sum_{i=0}^{N-1} a_i^2 \right) f''(\xi) = \frac{1}{2} C_N f''(\xi), \quad a < \xi < b.
\]

Proof. Dividing \( [a, b] \) into the subintervals \( [x_i, x_{i+1}] \) we obtain for the error \( R_N \)

\[
R_N = \sum_{i=0}^{N-1} \left( \int_{x_i}^{x_{i+1}} p(x)f(x)dx - \frac{1}{N} f(a_i) \right).
\]

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By the Taylor series

\[ f(x) = f(a_i) + (x - a_i)f'(a_i) + \frac{1}{2}(x - a_i)^2f''(\xi_i) \]

and by (3) we get the expression

\[
R_N = \frac{1}{2} \sum_{i=0}^{N-1} \left\{ \int_{x_i}^{x_{i+1}} (x - a_i)^2 p(x)f''(\xi_i)dx \right\}
\]

(6)

\[
= \frac{1}{2} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (x - a_i)^2 p(x)dx f''(\xi) .
\]

Furthermore, it follows from (3) that

\[
\sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (x - a_i)^2 p(x)dx = \sum_{i=0}^{N-1} \left\{ \int_{x_i}^{x_{i+1}} x^2 p(x)dx - \frac{2}{N} a_i^2 + \frac{1}{N} a_i \right\}
\]

(7)

\[
= \int_a^b x^2 p(x)dx - \frac{1}{N} \sum_{i=0}^{N-1} a_i^2 .
\]

(6) and (7) yield the bound (5).

\[ C_N \] can also be interpreted as the integration error of the function \( f = x^2 \). It may be noted that Jagermann's modification of the midpoint rule is obtained if the integral \( N \int_{y_N}^{(i+1)/N} L(y)dy \) in (3) is approximated by the (ordinary) midpoint rule, i.e., by \( L((2i + 1)/2N) \).

III. Examples.

(a) For \( p(x) = 1 \) and \( a = 0, b = 1 \), we obtain \( a_i = (2i + 1)/2N \) and, from (5), \( C_N = 1/12N^2 \) in accordance with the common midpoint rule.

(b) Let \( p(x) = \pi^{-1} (1 - x^2)^{-1/2} \) and \( a = -1, b = 1 \). From \( L(y) = -\cos \pi y \) it immediately follows that:

\[ a_i = -\frac{2N}{\pi} \sin \frac{\pi}{2N} \cos \frac{2i + 1}{2N} \pi \quad (i = 0, \ldots, N - 1) \]

and

\[
C_N = \frac{1}{2} - \frac{1}{N} \sum_{i=0}^{N-1} a_i^2 = \frac{1}{2} \quad \text{for } N = 1
\]

\[
= \frac{1}{2} - \frac{2N^2}{\pi^2} \sin \frac{\pi}{2N} \quad \text{for } N \geq 2 .
\]

Obviously, \( C_N = O(N^{-2}) \).

(c) For the infinite interval \( a = 0, b = \infty \) and the weight function \( p(x) = e^{-x} \) we get from \( L(y) = -\log (1 - y) \):

\[ a_{N-1} = 1 + \log N , \]

\[ a_i = 1 + \log N - (N - i - 1) \log (N - i - 1) - (N - i) \log (N - i) \]

for \( i = 0, 1, \ldots, N - 2 \). Numerically computed values of \( C_N \)
show that $C_N$ goes to 0 with the order $O(N^{-1})$.