Finite Difference Methods for the Computation of the "Poisson Kernel" of Elliptic Operators

By Pierre Jamet

1. Introduction. Most studies on numerical methods for elliptic differential equations have been devoted to the computation of bounded solutions. In this paper we study finite difference methods to compute an *unbounded* solution. The problem that we consider has been suggested by Professor J. L. Lions.

Let G be a bounded domain in R^2, with ∂G as its boundary. We assume that ∂G consists of a finite number of continuous closed curves. Let L be a differential operator of the form

$$Lu \equiv a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial y^2} + c \frac{\partial u}{\partial x} + d \frac{\partial u}{\partial y} - qu \,, \tag{1-1}$$

where the coefficients are functions of the point $P = (x, y) \in G$. We assume that these functions are Lipschitz-continuous in any interior subdomain of G, that is in every subdomain G' such that $\overline{G}' \subset G$. We also assume $a(P) > 0$, $b(P) > 0$ and $q(P) \geqq 0$ for all $P \in G$.

Let $Q_0 \in \partial G$ and $P_0 \in G$. We consider the differential problem

$$\begin{aligned} Lu(P) &= 0\,, \qquad P \in G\,, \\ u(P) &= 0\,, \qquad P \in \partial G - \{Q_0\}\,, \\ u(P_0) &= 1\,, \\ u(P) &> 0\,, \qquad P \in G\,, \\ u(P) &\in C^2(G) \cap C(\overline{G} - \{Q_0\})\,. \end{aligned} \tag{1-2}$$

We will construct a family of finite difference "approximations" and we will show that, under certain local conditions on the operator L near the boundary, this family contains a subsequence which converges to a solution of problem (1–2). This fact establishes the existence of a solution. Moreover, if we know that such a solution is unique,* we deduce that the whole family of our "approximations" converges to this unique solution; the convergence is uniform in $G - N(Q_0)$, where $N(Q_0)$ is an arbitrary neighborhood of Q_0.

The technique that we use in our proof is one which has already been used by the author and S. V. Parter [4], [6]; it is based on the notion of "discrete barrier" which goes back to I. G. Petrovsky [8]; a more recent and more general presentation of this technique can be found in [5].

In Section 2 we introduce the finite difference approximations and recall some useful results. In Section 3 we prove our existence and convergence theorem. In Section 4 we restrict our attention to operators with constant coefficients and we study the behavior of the approximations near the singularity. Finally, in Section 5

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* For instance, when G is a circle and L is the Laplacian, it is known that this solution is proportional to the Poisson kernel at the point Q_0 (see Rudin [9], problem 8, page 237). (The author is indebted to Professor S. V. Parter for this reference.)

we give an account of some numerical experiments; the author wishes to express his thanks to Mrs. F. Glain who carried out the computations.

2. Finite Difference Approximations. Let h be a vector in R^2 with positive components Δx and Δy. Let $R(h) = \{P = (i\Delta x, j\Delta y); i, j \text{ integers}\}$. For any point $P \in R(h)$, let $\mathfrak{N}(P) = \{P_1, P_2, P_3, P_4\} = \{((i \pm 1)\Delta x, (j \pm 1)\Delta y)\}$. To define a discrete analog of the domain G, we will use, for instance, approximation "of degree zero" (see [3]). That is, we define

$$G(h) = \{P \in R(h) \cap G; \mathfrak{N}(P) \in \overline{G}\} ,$$
$$\overline{G}(h) = G(h) \cup (\textstyle\bigcup_{P \in G(h)} \mathfrak{N}(P)) ,$$
$$\partial G(h) = \overline{G}(h) - G(h) .$$

We remark that for h small enough, $G(h)$ has the following "strong connectedness" property: for all $P \in G(h)$ and $Q \in \overline{G}(h)$, there exists a sequence of points $\{P_0, P_1, \cdots, P_n\}$ such that $P_0 = P$, $P_n = Q$, $P_i \in G(h)$ and $P_{i+1} \in \mathfrak{N}(P_i)$ for $0 \leqq i < n$.

Let L_h be a finite difference operator of the form

$$L_h v(P) = -A(P, P)v(P) + \sum_{Q \in \mathfrak{N}(P)} A(P, Q)v(Q) \tag{2-1}$$

where P denotes an arbitrary point of $G(h)$ and v an arbitrary function defined on $\overline{G}(h)$.

We assume that L_h is of positive type for h small enough, that is

$$\begin{aligned} &A(P, P) > 0 , \qquad A(P, Q) > 0 \quad \text{for all } P \in G(h) \text{ and all } Q \in \mathfrak{N}(P) , \\ &E(P) \equiv A(P, P) - \sum_{Q \in \mathfrak{N}(P)} A(P, Q) \geqq 0 . \end{aligned} \tag{2-2}$$

We assume also that L_h is a uniformly consistent approximation of order 1 to the differential operator L in any interior subdomain G', that is, given any $G' \subset \overline{G}' \subset G$ and any function $\phi(P) \in C^3(\overline{G}')$, $(L_h - L)\phi(P) = O(h)$ uniformly in G' as $h \to 0$.

The assumptions of Section 1 guarantee the existence of an operator L_h with such properties (see [5] where examples of such operators are given).

We will now make a further assumption on L_h, which will imply some conditions on the behavior of the functions $a(P)$, $b(P)$, $c(P)$, $d(P)$, $q(P)$ near the boundary. We assume that at each point $Q \in \partial G - \{Q_0\}$ there exists a local discrete barrier for the family of operators L_h that is, there exists a function $B(P, Q)$ and a neighborhood $N(Q)$ of Q such that

$$\begin{aligned} &B(P; Q) \in C(\overline{G} \cap N(Q)) , \\ &B(Q; Q) = 0 , \\ &B(P; Q) < 0 , \qquad \forall P \in \overline{G} \cap N(Q) - \{Q\} , \\ &L_h B(P; Q) - E(P) \geqq 0 , \\ &\text{for all} \quad P \in G(h) \cap N(Q) , \quad \text{and for all } h \text{ sufficiently small} . \end{aligned} \tag{2-3}$$

Local criterions which guarantee the existence of a local discrete barrier at Q can be found in [4], [5], [6]. In particular, if the operator L is uniformly elliptic and has bounded coefficients in G it is sufficient to assume that there exists a circle C whose intersection with $\overline{G}$ is the single point Q. However, we do *not* assume in general that L is uniformly elliptic nor has bounded coefficients in G.

Now let $Q_0(h) \in \partial G(h)$ and $P_0(h) \in G(h)$ be such that $Q_0(h) \to Q_0$ and $P_0(h) \to P_0$ as $h \to 0$.

Let us consider the problem

$$\begin{aligned} L_h v(P) &= 0 , \qquad P \in G(h) , \\ v(P) &= 0 , \qquad P \in \partial G(h) - \{Q_0(h)\} , \\ v(P_0(h)) &= 1 . \end{aligned} \tag{2-4}$$

This problem is a discrete analog of problem (1-2).

Before closing this section, we state two theorems which are trivial modifications of known results; these theorems will be used in the next section.

Let $\mathfrak{F}$ be a family of mesh functions $v(P, h)$ defined on $\overline{G}(h)$ for each h and such that $L_h v(P; h) = 0$ for all $P \in G(h)$. Let G' be an arbitrary interior subdomain of G; suppose h so small that G' is covered by square cells of the mesh; then, by linear interpolation in those cells, we can extend the definition of $v(P; h)$ to all G' so that $v(P; h) \in C(G')$. The following result holds.

THEOREM 2.1. *If the family $\mathfrak{F}$ is uniformly bounded in G, then it is equicontinuous in G'.*

Proof. This theorem is a slight modification of a theorem of W. V. Koppenfels [7], which is itself an extension of a theorem of Courant, Friedrichs and Lewy [2] for the Laplace equation. It is easy to show that our consistency assumption is equivalent to the requirement that the operator L_h has the form

$$L_h v = a' v_{x\bar{x}} + b' v_{y\bar{y}} + c' \frac{v_x + v_{\bar{x}}}{2} + d' \frac{v_y + v_{\bar{y}}}{2} - q' v$$

where v_x', $v_{\bar{x}}'$, $\cdots$ denote the usual forward and backward difference quotients of v and where

$$\begin{aligned} a' &= a'(P; h) = a(P) + O(h) , \\ b' &= b'(P; h) = b(P) + O(h) , \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ q' &= q'(P; h) = q(P) + O(h) , \end{aligned} \tag{2-5}$$

uniformly in any interior subdomain for h small. Now, conditions (2-5) together with the Lipschitz-continuity of the coefficients $a(P)$, $\cdots$, $q(P)$ in interior subdomains imply the validity of Koppenfels' result on equicontinuity of the family $\mathfrak{F}$ in G'.**

Now, let $\partial^{(1)}G$ and $\partial^{(2)}G$ be two complementary subsets of ∂G. We assume that at each point Q of $\partial^{(1)}G$ there exists a local discrete barrier for the family of operators L_h. Let $\partial^{(1)}G(h)$ be the set of those points in $\partial G(h)$ whose distance to $\partial^{(1)}G$ is less than h. Let $g(P) \in C(\overline{G})$ and let $\mathfrak{F}$ be a family of functions $v(P; h)$ which satisfy, for each h:

** Koppenfels stated this result under somewhat different conditions: he considers a more general type of operator, but his assumptions on the coefficients are stronger; also, he is interested in the equicontinuity of the first and second difference quotients of the functions $v(P; h)$. It is easy to check that our assumptions are sufficient.

$$(2\text{-}6)\qquad \begin{aligned} L_h v(P;h) &= 0\,, \qquad P \in G(h)\,,\\ v(P;h) &= g(P)\,, \qquad P \in \partial^{(1)}G(h)\,. \end{aligned}$$

The following result holds.

THEOREM 2.2. *Assume the family* $\mathfrak{F}$ *is uniformly bounded. Then, it admits a subsequence* $\{v(P;h_n);\, h_n \to 0\}$ *which converges to a function* $u(P)$ *which satisfies:*

$$(2\text{-}7)\qquad \begin{aligned} Lu(P) &= 0\,, \qquad P \in G\,,\\ u(P) &= g(P)\,, \qquad P \in \partial^{(1)}G\,,\\ u(P) &\in C^2(G) \cap C(G \cup \partial^{(1)}G)\,. \end{aligned}$$

The convergence is uniform in $G - N$ *where* N *is an arbitrary neighborhood of* $\partial^{(2)}G$, *i.e.,*

$$\operatorname*{Max}_{P \in G(h) \cap (G-N)} |v(P;h) - u(P)| \to 0 \quad \text{as } h \to 0\,.$$

Proof. This theorem is a trivial modification of Theorems 2-1 and 2-2 of [6]. A complete proof can be found in [5]; this proof assumes interior equicontinuity of the family $\mathfrak{F}$ and, therefore, Theorem 2.1 is needed.

Remark. The particular case $\partial^{(1)}G = \partial G$ is of special interest. In that case, conditions (2-7) imply unicity of the limit function $u(P)$ and therefore, the whole family $\mathfrak{F}$ converges to $u(P)$ as $h \to 0$; the convergence is uniform in G.

3. Existence and Convergence Theorem.

LEMMA 3.1. *For* h *small enough, problem* (2-4) *has a unique solution* $v(P;h)$ *defined on* $\overline{G}(h)$.

Proof. Let $z(P)$ be a function defined on $\overline{G}(h)$ which satisfies the homogeneous system corresponding to (2-4), i.e.

$$(3\text{-}1)\qquad \begin{aligned} L_h z(P) &= 0\,, \qquad P \in G(h)\,,\\ z(P) &= 0\,, \qquad P \in \partial G(h) - \{Q_0(h)\}\,,\\ z(P_0(h)) &= 0\,. \end{aligned}$$

Let $z_0 = z(Q_0(h))$ and suppose $z_0 > 0$. Since $G(h)$ is "strongly connected" for h small enough and L_h is of positive type, we can apply the "strict" maximum principle and deduce $0 < z(P) < z_0$ for all $P \in G(h)$. This contradicts the fact that $z(P_0(h)) = 0$; therefore, $z_0 \leqq 0$. Similarly we deduce $z_0 \geqq 0$ and hence $z_0 = 0$. This implies: $z(P) = 0$ for all $P \in G(h)$ and the lemma follows at once. Moreover, we note that

$$0 < v(P;h) < v(Q_0(h);h) \quad \text{for all } P \in G(h)\,.$$

In the following we will always assume h so small that problem (2-4) has a unique solution and we will denote by $S = \{v(P;h_n);\, h_n \to 0\}$ a sequence of those solutions.

LEMMA 3.2. *Let* $N(Q_0)$ *be an arbitrary neighborhood of* Q_0 *in* R^2. *Then, the sequence* S *is uniformly bounded in* $G - N(Q_0)$.

Proof. Suppose S is not uniformly bounded in $G - N(Q_0)$. Then, for every $M > 0$, there exists an infinite subsequence $S_M \subset S$ such that

$$(3\text{-}2)\qquad \operatorname*{Max}_{P \in G(h) - N(Q_0)} v(P;h) > M \quad \text{for all } v \in S_M$$

In the following, we consider only functions $v(P;h)$ in S_M. Using the maximum principle, we deduce that, for each h, there exists a finite sequence of points $L(h) = \{P_1, P_2, \cdots, P_n\}$ such that

$$\begin{aligned} &P_1 \in G(h) - N(Q_0)\,, \\ &P_i \in G(h)\,, \qquad i = 1, 2, \cdots, (n-1)\,, \\ &P_{i+1} \in N(P_i)\,, \\ &P_n = Q_0(h)\,, \\ &v(P_i; h) > M\,, \qquad i = 1, 2, \cdots, n\,. \end{aligned} \tag{3-3}$$

Let N' and N'' be two open neighborhoods of Q_0 in R^2 with smooth boundaries and such that

(i) $\overline{N}'' \subset N' \subset N(Q_0)$,

(ii) $\partial G \cap (N' - \overline{N}'')$ consists of two disjoint connected subsets of ∂G, say Γ_1 and Γ_2 (see Fig. A).

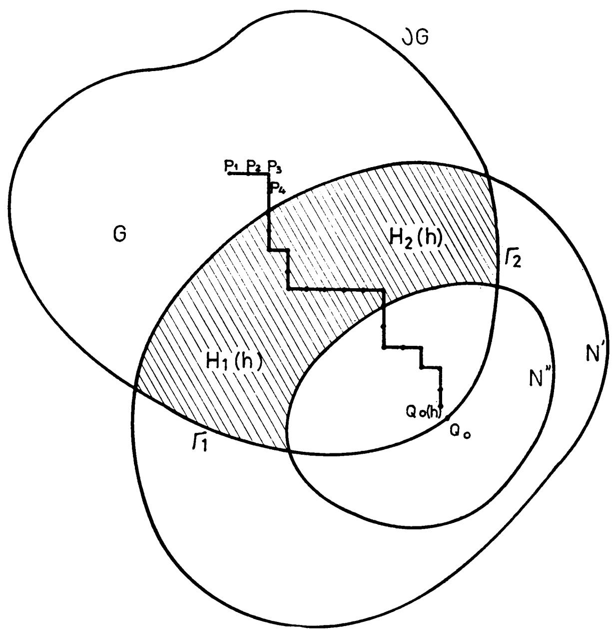

FIGURE A

Let $G_0 = G \cap (N' - \overline{N}'')$. For any subdomain G' of G with boundary $\partial G'$, we define discrete sets $G'(h)$, $\partial G'(h)$ and $\overline{G}'(h)$ in the same way we have defined $G(h)$, $\partial G(h)$ and $\overline{G}(h)$. In particular, we consider now the set $\overline{G}_0(h)$. Suppose h so small that $Q_0(h) \in N''$. Then, we have

$$\overline{G}_0(h) = H_1(h) \cup L_0(h) \cup H_2(h)\,, \tag{3-4}$$

where $L_0(h) = L(h) \cap \overline{G}_0(h)$ and where $H_1(h)$, $H_2(h)$ are the two subsets of $\overline{G}_0(h)$ lying "on each side" of $L_0(h)$ (say $H_1(h)$ is on the same side as Γ_1). Let $s = 1$ or 2 and let $g_s(P) \in C(\overline{G}_0)$ be such that $0 \leqq g_s(P) \leqq 1$ in G_0, $g_s(P) \equiv 0$ in a neighborhood of $\partial G_0 - \Gamma_s$ and $g_s(P) \not\equiv 0$ on Γ_s. Let $v_s(P, h)$ be the solution of the problem

$$(3\text{-}5)\qquad \begin{aligned} L_h v_s(P) &= 0\,, \qquad P \in G_0(h)\,,\\ v_s(P) &= g_s(P)\,, \quad P \in \partial G_0(h)\,. \end{aligned}$$

It follows from the maximum principle that, for h small enough

$$(3\text{-}6)\qquad v(P;h) > Mv_s(P;h)\,, \qquad \forall P \in H_{s'}(h) \cup L_0(h)\,,$$

where $s' = 1$ or 2; $s' \neq s$. Therefore, $v(P;h) > M \min_{s=1,2} v_s(P;h)$, $\forall P \in G_0(h)$ because of (3-4). But, applying Theorem 2.2 in the domain G_0, we deduce that $v_s(P;h)$ converges uniformly in G_0 as $h \to 0$ to a function $u_s(P)$ which is strictly positive in any interior subdomain G_0' of G_0.

Let

$$c_0 = \inf_{P \in G_0'} \left\{ \min_{s=1,2} u_s(P) \right\}.$$

For h small enough, we have

$$(3\text{-}7)\qquad v(P;h) > Mc_0/2\,, \qquad \forall P \in G_0'(h)\,.$$

Now, let G' be a smooth interior subdomain of G, with boundary $\partial G'$, such that $P_0 \in G'$ and $\partial G' \cap G_0' \neq \varnothing$. Let $\Gamma_3 = \partial G' \cap G_0'$ and let $g_3(P) \in C(\overline{G}')$ be such that $P_0 \in G'$ and $\partial G' \cap G_0' \neq \varnothing$. Let $\Gamma_3 = \partial G' \cap G_0'$ and let $g_3(P) \in C(\overline{G}')$ be such that $0 \leqq g_3(P) \leqq 1$ in G', $g_3(P) \equiv 0$ in a neighborhood of $\partial G' - \Gamma_3$ and $g_3(P) \not\equiv 0$ on Γ_3. Let $v_3(P;h)$ be the solution of the problem

$$(3\text{-}8)\qquad \begin{aligned} L_h v_3(P;h) &= 0\,, \qquad P \in G'(h)\,,\\ v_3(P;h) &= g_3(P)\,, \qquad P \in \partial G'(h)\,. \end{aligned}$$

It follows from Theorem 2.2 that $v_3(P;h)$ converges uniformly in G' as $h \to 0$ to some function $u_3(P)$ which is strictly positive in any interior subdomain G'' of G'.

Choose G'' such that $P_0 \in G''$ and let $c_1 = \inf_{P \in G''} u_3(P)$. For h small enough, we have $P_0(h) \in G''(h)$ and $v_3(P;h) > c_1/2$, $\forall P \in G''(h)$; therefore

$$(3\text{-}9)\qquad v_3(P_0(h);h) > c_1/2\,.$$

Using (3-7) and (3-9) and applying the maximum principle, we deduce that, for h small enough,

$$(3\text{-}10)\qquad v(P_0(h);h) > M(c_0/2)(c_1/2)\,.$$

But $v(P_0(h);h) = 1$ by (2-4) and M is arbitrarily large; therefore, we have reached a contradiction and the lemma is proved.

THEOREM 3-1. *Let* $S = \{v(P;h_n);\, h_n \to 0\}$ *be an arbitrary sequence of solutions of* (2-4). *Then, S admits a subsequence which converges to a solution of problem* (1-2); *the convergence is uniform in* $G - N(Q_0)$, *where* $N(Q_0)$ *is an arbitrary neighborhood of* Q_0. (*Moreover, if the solution of problem* (1-2) *is unique, the whole sequence S converges to this solution.*)

Proof. We will assume that $N(Q_0)$ is open and $P_0 \notin N(Q_0)$. Let N' be a neighborhood of Q_0 such that $\overline{N}' \subset N(Q_0)$. The sequence S is uniformly bounded in $G - N'$ by the preceding lemma. It follows from Theorem 2.2 that there exists a subsequence S_0 of S which converges uniformly in $G - N(Q_0)$ to a function $u(P)$ with the following properties:

$$(3\text{-}11)\qquad \begin{aligned} &Lu(P) = 0\,, \qquad P \in G - N(Q_0)\,,\\ &u(P) = 0\,, \qquad P \in \partial G - N(Q_0)\,,\\ &u(P_0) = 1\,,\\ &u(P) > 0 \quad \text{in } G - N(Q_0)\,,\\ &u(P) \in C^2(G - N(Q_0)) \cap C(\overline{G} - N(Q_0))\,. \end{aligned}$$

Now, let us consider a decreasing sequence $\{N_r(Q_0)\}$ of neighborhoods of Q_0 such that $\bigcap_{r=1}^{\infty} N_r(Q_0) = \{Q_0\}$. By taking successive refinements of the subsequence S_0 we can extend recursively the definition of the function $u(P)$ in $G - N_1(Q_0)$, $G - N_2(Q_0)$, $\cdots$ and finally, in $N - \{Q_0\}$ by using a diagonal procedure. The extended function is a solution of problem (1-2).

The rest of the theorem follows at once.

Remark 3-1. The results of this section are also valid for other types of approximation near the boundary (not only for approximation of degree zero).

Remark 3-2. It is expected that Theorem 3-1 is also valid in R^n, $n > 2$. However, our proof of Lemma 3.2 cannot be extended to more than two dimensions.

4. Estimates Near Singularity. In this section, we assume that the operator (1-1) and its discrete analog (2-1) have constant coefficients. For greater simplicity we assume $\Delta x = \Delta y$ and we define $h = \Delta x = \Delta y$. We will assume the uniqueness of the solution of problem (1-2).

THEOREM 4-1. *Assume that G is convex in a neighborhood of Q_0 and that there exists a constant K, $0 < K < 1/\sqrt{2}$ such that*

$$(4\text{-}1)\qquad d(Q_0(h), Q_0) < Kh\,.$$

Then, for h small enough, the following inequality holds:

$$(4\text{-}2)\qquad v(Q_0(h); h) > c/h\,,$$

where c is some positive constant (independent of h).

Proof. First, we introduce the following notations: Given any point P in R^2 and any positive number ρ, we denote by $S(P;\rho)$ the open sphere with center P and radius ρ. Given any set $E \subset R^2$ and any couple of points P and P' in R^2, we denote by $E_{PP'}$ the set deduced from E by the translation $P \to P'$.

It follows from the local convexity of G at Q_0 that there exists a straight line D through Q_0 and a sphere $S(Q_0;\rho)$ such that $G \cap S(Q_0;\rho)$ lies entirely in one of the two half-planes separated by D; let H be this half-plane. Let us choose ρ so small that

$$(4\text{-}3)\qquad S(P_0;\rho) \subset G\,.$$

Let $T = H \cap S(Q_0;\rho/2)$ and $G_1 = \bigcup_{P\in T} G_{Q_0P}$. It follows from these definitions that $D \cap S(Q_0;\rho/2) \subset \partial G_1$. Now, let $\Gamma(h)$ be the set of all points $P \in \partial G_1(h) \cap S(Q_0;\rho/2)$ such that $[\overline{G}(h)]_{Q_0(h)P} \subset \overline{G}_1(h)$.

Let $\nu(h)$ be the number of points in $\Gamma(h)$. It follows from (4-1) that there exists a constant $K_1 > 0$ such that, for h small enough

$$(4\text{-}4)\qquad \nu(h) > K_1/h\,.$$

For each h and each $Q \in \Gamma(h)$, let $v_1(P; h, Q)$ be the solution of the problem

$$(4\text{-}5)\qquad \begin{aligned} L_h v_1(P) &= 0\,, && P \in G_1(h)\,, \\ v_1(Q) &= 1\,, \\ v_1(P) &= 0\,, && P \in \partial G_1(h) - \{Q\} \end{aligned}$$

(note that it is trivial to extend the definition of the operator L_h on $G_1(h)$ since, by assumption, this operator has constant coefficients).

Let $v_0(P; h) = v(P; h)/v(Q_0(h); h)$. It follows from (2-4) that $v_0(P; h)$ satisfies

$$(4\text{-}6)\qquad \begin{aligned} L_h v_0(P) &= 0\,, && P \in G(h)\,, \\ v_0(Q_0(h)) &= 1\,, \\ v_0(P) &= 0\,, && P \in \partial G(h) - \{Q_0(h)\}\,. \end{aligned}$$

Let $P' = [P_0(h)]_{QQ_0(h)}$. Since $Q \in \Gamma(h)$, we have $[\overline{G}(h)]_{Q_0(h)Q} \subset \overline{G}_1(h)$ and therefore, applying the maximum principle, we deduce

$$(4\text{-}7)\qquad v_1(P_0(h); h, Q) \geqq v_0(P'; h) = v(P'; h)/v(Q_0(h); h)\,.$$

But, for h small enough, $P' \in S(P_0;\, (3/4)\rho) = G^* =$ fixed interior subdomain of G, because of (4-3). By Theorem 3-1 and because of our uniqueness assumption on the solution of problem (1-2), $v(P; h)$ converges uniformly in G^* as $h \to 0$ to a function which is strictly positive in G^*; therefore, there exists a constant K_2 such that:

$$(4\text{-}8)\qquad v(P'; h) > K_2 > 0 \qquad \text{for } h \text{ small enough}\,.$$

On the other hand, the function $w(P; h) \equiv \sum_{Q \in \Gamma(h)} v_1(P; h, Q)$ satisfies

$$\begin{aligned} L_h w(P) &= 0\,, && P \in G_1(h)\,, \\ w(P) &= 1\,, && P \in \Gamma(h) \subset \partial G_1(h)\,, \\ w(P) &= 0\,, && P \in \partial G_1(h) - \Gamma(h)\,, \end{aligned}$$

and, therefore, the maximum principle implies

$$\sum_{Q \in \Gamma(h)} v_1(P; h, Q) \leqq 1\,, \qquad \forall P \in G_1(h)\,.$$

Using this inequality together with (4-7), (4-8) and (4-4) we deduce

$$(4\text{-}10)\qquad 1 > \nu(h)\,\frac{K_2}{v(Q_0(h); h)} > \frac{1}{h}\,\frac{K_1 K_2}{v(Q_0(h); h)}\,,$$

which ends the proof of the theorem.

Now, we state two direct corollaries of Theorems 3-1 and 4-1. They involve the function $v_0(P; h)$ which is the unique solution of problem (4-6). Such a function has been considered (with different notations) by many authors, in particular by Courant-Friedrichs and Lewy [2] and by Bramble and Hubbard [1]. However, the following results seem to be new.

COROLLARY 4-1. *Let G and $Q_0(h)$ satisfy the hypotheses of Theorem* 4-1. *Let N be an arbitrary neighborhood of Q_0 and let $v_0(P; h)$ be the unique solution of problem* (4-6). *Then, there exists a positive constant c_0 such that, for h small enough*

$$(4\text{-}11)\qquad 0 < v_0(P; h) < c_0 h \quad \textit{for all } P \in G(h) - N\,.$$

COROLLARY 4-2. *Let G and $Q_0(h)$ satisfy the hypotheses of Theorem* 4-1. *Let $V(P; h) = (1/h)v_0(P; h)$, where $v_0(P; h)$ is the unique solution of problem* (4-6).

Then every sequence $\{V(P; h_n); h_n \to 0\}$ *admits a subsequence which converges to a function* $U(P)$ *which is proportional to the solution* $u(P)$ *of problem* (1-2).

However, it must be noted that $U(P)$ *may be identically zero and that the sequence itself does not converge in general.*†

Proof. It follows from Theorem 4-1 and Lemma 3.2 that the family of functions $\{V(P; h)\}$ is uniformly bounded in $G - N(Q_0)$, where $N(Q_0)$ is an arbitrary neighborhood of Q_0. Therefore, by the same argument as for Theorem 3-1, we deduce the existence of a converging subsequence. The limit function satisfies the conditions

$$\begin{aligned} Lu(P) &= 0\,, && P \in G\,, \\ u(P) &= 0\,, && P \in \partial G - \{Q_0\}\,, \\ u(P) &\geqq 0\,, && P \in G\,, \\ u(P) &\in C^2(G) \cap C(\overline{G} - \{Q_0\})\,. \end{aligned}$$

It may be any nonnegative function which is proportional to the solution of problem (1-2).

Remark 4-1. The condition (4-1) can be easily weakened. For instance, let $d_x(Q_0(h), D)$ denote the "horizontal distance" from $Q_0(h)$ to D, i.e., the distance between $Q_0(h)$ and the intersection of D with the straight line through $Q_0(h)$ parallel to the x-axis. In the same way, let $d_y(Q_0(h), D)$ denote the "vertical distance" from $Q_0(h)$ to D. A look at the proof of Theorem 4-1 shows that it is sufficient to assume that there exists a line D defined as before such that

$$\min\{d_x(Q_0(h), D), d_y(Q_0(h), D)\} < K'h \tag{4-12}$$

where K' is some constant, $0 < K' < 1$.

Remark 4-2. If we assume that the domain G is concave in a neighborhood of Q_0, it is easy to prove, by the same kind of argument as for Theorem 4-1, that

$$v(Q_0(h); h) < c'/h\,. \tag{4-13}$$

(Instead of the domain G_1 we must now introduce a domain $G_2 = \bigcap_{P \in T} G_{Q_0 P}$, for some suitably defined set T.)

5. Numerical Experiments. (a) We take $L = \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and we consider the two following examples.

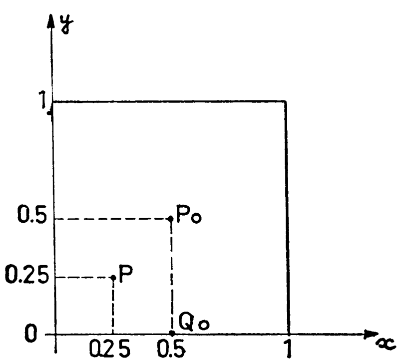

FIGURE 1

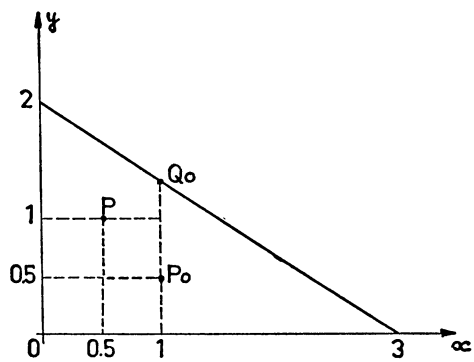

FIGURE 2

† See Section 5: Numerical experiments.

Example 1. G is the unit square shown on Fig. 1 and $Q_0 = (1/2, 0)$, $P_0 = (1/2, 1/2)$. We will consider, for example, the point $P = (1/4, 1/4)$.

Example 2. G is the triangle shown on Fig. 2 and $Q_0 = (1, 4/3)$, $P_0 = (1, 1/2)$. We will consider, for example, the point $P = (1/2, 1)$.

In both cases, we take $h = \Delta x = \Delta y = 1/N = 2^{-n}$, n integer.

Hence, in the first example, we have $Q_0 \in \bar{G}(h)$, $P_0 \in G(h)$, $\partial G(h) \subset \partial G$. But, in the second example, $Q_0 \notin \bar{G}(h)$ and $\partial G(h) \not\subset \partial G$. In the first example we choose $Q_0(h) = Q_0$, $P_0(h) = P_0$ and in the second example we choose $Q_0(h) =$ the point of $\partial G(h)$ which is the closest to Q_0, $P_0(h) = P_0$.

In both cases, L_h is the usual five-point approximation of the Laplacian and we consider the functions $v(P; h_n)$ and $V(P; h_n)$ of Theorem 3-1 and of Corollary 4-2.

In Tables I and II, we give the values of those functions at the point P; Table I corresponds to the first example and Table II corresponds to the second example.

TABLE I (*Square*)

$N = 1/H$	$v(P, H)$	$V(P, H)$
4	0.7857	0.3928
8	1.0252	0.4523
16	1.1257	0.4755
32	1.1528	0.4825
64	1.1598	0.4843
128	1.1616	0.4848

TABLE II (*Triangle*)

$N = 1/H$	$v(P, H)$	$V(P, H)$
2	0.9375	0.5741
4	1.6787	0.5214
8	1.6267	0.6321
16	1.8765	0.4935
32	1.8208	0.6332
64	1.8752	0.4813

We observe that, in both cases, $v(P; h_n)$ converges as n increases; but the convergence is faster in the first case (a closer examination shows that the convergence is $O(h^2)$ in this case). On the other hand, $V(P; h_n)$ converges only in the first case; in the second case, it seems that the corresponding sequence has two limit points (see Fig. 3); the difference between these two cases comes of course from the fact that, in the second case, $\partial G(h) \not\subset \partial G$ and $Q_0(h) \neq Q_0$.†† These results are in agreement with Theorem 3-1 and Corollary 4-2.

(b) Now we check the conclusion of Theorem 4-1.

Example 3. Same as Example 1 except that $Q_0 = (0, 0) =$ the origin.

In this case $\partial G(h) \subset \partial G$, but $Q_0 \notin \partial G(h)$ and therefore, we cannot choose $Q_0(h)$

†† In that case it would be easy to choose the mesh so that $\partial G(h) \subset \partial G$ and $Q_0(h) = Q_0$. For a general domain in R^2, one should use another type of approximation near the boundary ("full grid approximation"; see [1], [3], [6]).

$= Q_0$; we choose $Q_0(h) = (h, 0)$. The condition (4-12) is satisfied, since we can take the x-axis for D, and thus we have $d_y(Q_0(h), D) = 0$. Therefore, by Theorem 4-1, we must have $v(Q_0(h); h) > ch^{-1}$.

TABLE III

$N = 1/H$	$v(Q_0(H), H)$	$\beta(H)$
4	$0.16000E + 01$	
8	$0.60444E + 02$	1.917
16	$0.23614E + 03$	1.966
32	$0.93809E + 03$	1.990
64	$0.37456E + 04$	1.997

In Table III we give the values of $v(Q_0(h_n); h_n)$, and we compute

$$\beta(h) = \frac{1}{\log 2} \log \frac{v(Q_0; h)}{v(Q_0; 2h)} . \tag{5-1}$$

We observe that $\beta(h) \to \beta = 2$ as h decreases, which shows that

$$v(Q_0(h); h) \sim ch^{-2} > ch^{-1} . \tag{5-2}$$

Example 4. As a generalisation of Example 3 we consider the domain shown on Fig. 4 with $\theta = \pi/4, \pi/2, 3\pi/4, \cdots, 2\pi$. We compute $\beta(h)$ as in Example 3 and we

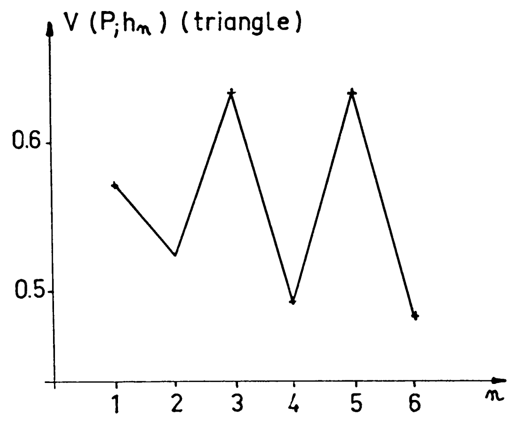

FIGURE 3

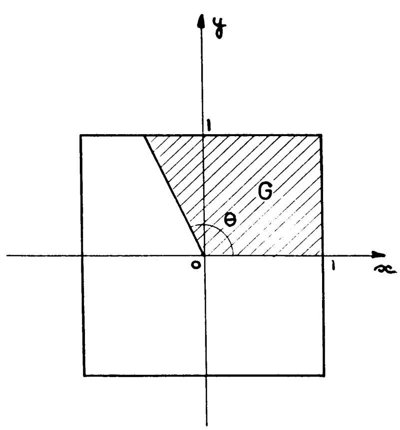

FIGURE 4

observe that $\beta(h)$ converges to $\beta = \pi/\theta$ as h decreases which shows that

$$v(Q_0(h); h) \sim ch^{-\pi/\theta} . \tag{5-3}$$

Therefore,

$$v(Q_0(h); h) > c'h^{-1} \quad \text{if } \theta \leqq \pi \text{ (convex case)} ,$$
$$v(Q_0(h); h) < c''h^{-1} \quad \text{if } \theta \geqq \pi \text{ (concave case)} .$$

Finally, Fig. 5 gives a representation of the solution in the case of Example 2 (triangle).

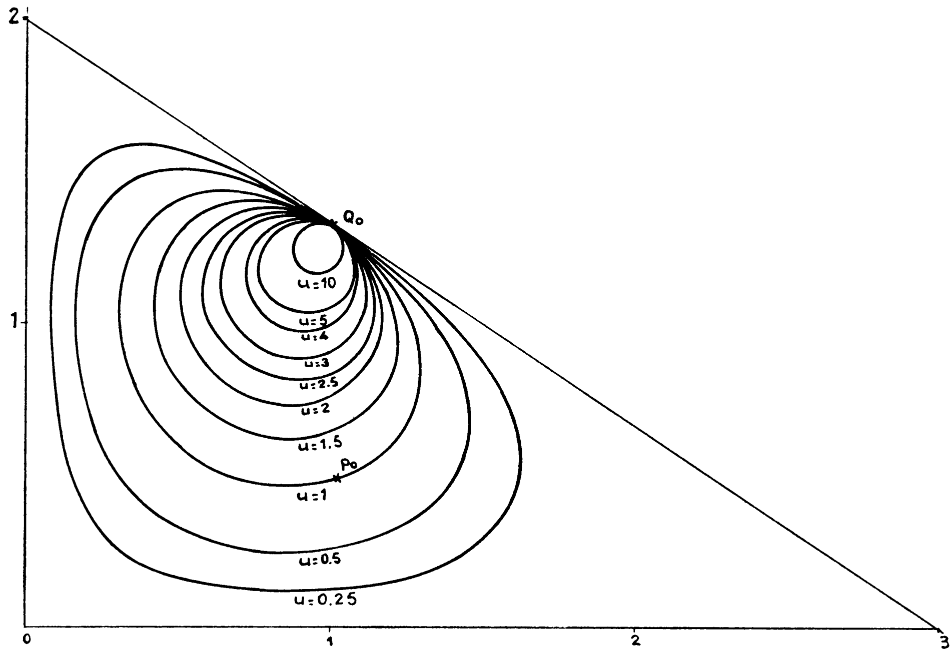

FIGURE 5

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