

## REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

61[2.05, 2.20, 2.35, 3, 4, 5].—F. JOHN, *Lectures on Advanced Numerical Analysis*, Gordon and Breach, New York, 1967, xiv + 179 pp., 24 cm. Price \$9.50 cloth bound, \$6.25 paper bound.

This improved version of Fritz John's New York University lecture notes, entitled *Advanced Numerical Methods* and first issued in 1956, has been long awaited.

From the treatment of norms of vectors and matrices to the study of methods in linear algebra, in approximation theory, in finding roots of systems of equations, in solving ordinary and partial differential equations, the author leaves his characteristic imprint. Here is the work of a master analyst who spent an intensive and short productive period in the illumination of a wide variety of topics in numerical analysis. The book is a classic that invites mathematicians to learn what numerical analysis is about. The reviewer is only one of the generations that have benefitted from this fine work.

E. I.

62[3, 10, 13.35].—A. KAUFMANN, *Graphs, Dynamic Programming, and Finite Games*, translated from French by Henry C. Sneyd, Academic Press, New York, 1967, xvii + 484 pp., 24 cm. Price \$14.50.

This volume is an attempt to introduce at both an elementary level and a mathematically rigorous level three areas of modern mathematics—graph theory, dynamic programming, and game theory. To do this, the author divides the presentation into two parts, each part having three chapters, one for each topic. The first part has elementary applications of each theory, which are worked out in detail. The second part is a detailed mathematical treatment of each area.

Typical of the first part of the book are worked examples, such as critical paths, minimal trees, network flows and textual emendation problems, which are solved step-by-step in the graph chapter, with extensive diagrams and numerical calculations. In these elementary chapters concepts are introduced quickly and effectively with a minimum of definitional complexity and immediate appeal to easily understood examples.

The second part of the work is an attempt to treat in less than 100 pages per topic the mathematical theory of each of the three areas. The graph theory chapter in this part of the book paraphrases much of the material in Berge [1], but in addition contains detailed descriptions of an algorithm of the author's for finding Hamilton paths, and a section on the graph isomorphism problem. The game theory chapter follows McKinsey [2] in its proof of the fundamental theorem of game theory on the existence of optimal strategies for rectangular games. Various examples are worked (also many examples are worked in the elementary chapter on game theory) and the correspondence to linear programming is treated along with brief presentations of other topics. The dynamic programming chapter has material which appears in more detail in Kaufmann and Cruon [3] in another volume of the same series. Of

special interest is the author's use of the zeta-transform to study Markovian processes.

In attempting to treat so much material in the advanced portion of the book, the author is not able to maintain the same clarity found in the first part of the book. Some topics are explained thoroughly in the advanced part, but others are poorly developed and confusing. I count the first half a great success as a lucid introduction to these topics. However, the second half is more useful when augmented by readings from other sources. A minor irritation is an abundance of typographical errors. Its use as a text is possible for many types of courses; it has a useful bibliography and index but no exercises. It is a wide-ranging book with sections comprehensible at many levels of mathematical sophistication.

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1. CLAUDE BERGE, *The Theory of Graphs and Its Applications*, Methuen, London, 1962.
2. J. C. C. MCKINSEY, *An Introduction to the Theory of Games*, McGraw-Hill, New York, 1952.
3. A. KAUFMANN & R. CRUON, *Dynamic Programming*, Academic Press, New York, 1967.

**63[4].**—WERNER GLASMACHER & DIETMAR SOMMER, *Implizite Runge-Kutta-Formeln*, Westdeutscher Verlag, Köln, 1966, 178 pp., 24 cm. Price DM 44.00.

Implicit Runge-Kutta methods based on Gauss-Legendre quadrature formulae were introduced by Ceschino and Kuntzmann [1] and by the reviewer [2]. Methods based on Lobatto and on Radau quadrature formulae were introduced by the reviewer [3]. These methods have the property that if  $m$  is the number of stages, then the order is  $2m$  for the Gauss case,  $2m - 2$  for the Lobatto case, and  $2m - 1$  for the two types of the Radau case. A disadvantage of the methods for integrating differential equations in practice is their implicit nature.

The only previous tables of the coefficients of the methods are those of the reviewer [4], which give coefficients to 20D for methods of the four types with orders not exceeding 20. The present tables give coefficients to 24S for the four methods up to  $m = 20$ . In addition to the tables, full descriptions of the evaluation methods are given, including an Algol programme and flow charts.

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1. F. CESCINO & J. KUNTZMANN, *Problèmes Différentiels de Conditions Initiales*, Dunod, Paris, 1963.
2. J. C. BUTCHER, "Implicit Runge-Kutta processes," *Math. Comp.*, v. 18, 1964, pp. 50-64.
3. J. C. BUTCHER, "Integration processes based on Radau quadrature formulas," *Math. Comp.*, v. 18, 1964, pp. 233-244.
4. J. C. BUTCHER, *Tables of Coefficients for Implicit Runge-Kutta Processes*, ms. of 9 sheets deposited in the UMT file [see *Math. Comp.*, v. 19, 1965, p. 348, RMT 56].

**64[4].**—MINORU URABE, *Nonlinear Autonomous Oscillations, Analytical Theory*, Academic Press, New York, 1967, xi + 330 pp., 24 cm. Price \$16.00.

This monograph contains results in the theory of nonlinear autonomous oscillations, most of them based on the author's research. The main topic is the analytical

and numerical construction of periodic solutions. In attempting "to present a self-contained and readable account for mathematicians, physicists, and engineers" the author includes the statement of many basic theorems of ordinary differential equation theory and, with the aid of a mysterious selection principle, some proofs. As a consequence, a careful nonspecialist must have access to a modern textbook on ordinary differential equations; hence, he does not need at least half the monograph. Since the specialist will proceed directly to the more advanced topics, the supplementary material is presumably intended for those who want to refer to the theorems and need to see enough proofs so that they can believe that a complete theory exists.

The material most likely to be of interest to numerical analysts is included in a chapter on the numerical computation of periodic solutions and in an appendix on Newton's method and numerical methods for solving ordinary differential equations. In the chapter the author reduces the search for the initial values of a periodic solution of an  $n$ th order system of ordinary differential equations to the solution by Newton's method of a system of  $n - 1$  algebraic equations. The functions occurring in the algebraic equations are evaluated by integrating the differential equations written relative to a moving orthogonal coordinate system. The problem is nontrivial and is carefully treated. Several numerical examples and graphs are included. In addition, the Galerkin procedure is briefly discussed.

In general, the book is carefully written and can be used as a supplementary text for a course in ordinary differential equations or numerical analysis. The students should be warned, however, that the author sometimes uses inappropriate mathematical formalisms in heuristic discussions. For example, on p. 298 in a discussion of when to terminate an iterative scheme in a practical computation, we are told to take  $\epsilon = o(\alpha)$ , where  $\epsilon$  is the given bound on the error and  $\alpha$  is the cutoff criterion. On the other hand, in mathematical discussions, the terms "small," "accurate" and "approximate" are used most casually. For example, on p. 283 in a discussion of an iterative solution to  $x = f(x)$ , he writes "if [the starting approximate solution]  $x_0$  is accurate, then the quantity  $|f(x_0) - x_0| = |x_1 - x_0|$  is small." One is reminded of the ancient joke, Q. "How's your wife?" A. "Compared to what?"

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65[7].—CHIH-BING LING, *On the Values of Two Coefficients related to the Weierstrass Elliptic Functions*, Virginia Polytechnic Institute, Blacksburg, Virginia, January 1968, ms. of 4 typewritten sheets deposited in UMT file.

Using Jacobi theta functions, the author has herein extended to 101S the results of his previous calculation [1] of the following two coefficients, which are related to the Weierstrass functions of double periods  $(1, i)$  and  $(1, e^{\pi i/3})$ , respectively:

$$\sigma_4 = 3.15121\ 20021\ 53897\ 53821\ 76899\ 42248\ 68855\ 66455\ 19354\ 51485 \\ 24384\ 70540\ 35738\ 42598\ 37682\ 74612\ 16108\ 69439\ 55074\ 50822,$$

$$\sigma_6 = 5.86303\ 16934\ 25401\ 59797\ 02134\ 43837\ 82343\ 75153\ 76204\ 12955 \\ 75122\ 82731\ 11230\ 49523\ 95831\ 56859\ 89351\ 55366\ 27614\ 95871.$$

Also tabulated here to 101S are decimal approximations to  $e^{\pm\pi^{1/2}}$ ,  $e^{\pm\pi\sqrt{3}/2}$ ,  $K(\sin 45^\circ)$ , and  $K(\sin 15^\circ)$ .

Comparison of the last two constants with unpublished values by J. W. Wrench, Jr. to 164D and 77D, respectively, has revealed no discrepancies.

#### AUTHOR'S SUMMARY

1. CHIH-BING LING, "Evaluation at half periods of Weierstrass' elliptic functions with double periods 1 and  $e^{i\alpha}$ ," *Math. Comp.*, v. 19, 1965, pp. 658-661.

66[7].—OSCAR L. FLECKNER, *Table of Values of the Fresnel Integrals*, ms. of 8 pp. deposited in the UMT file.

This manuscript table consists of 6D values of the Fresnel Integrals  $(2\pi)^{-1/2} \int_0^x t^{-1/2} \cos t \, dt$  and  $(2\pi)^{-1/2} \int_0^x t^{-1/2} \sin t \, dt$ , which are generally designated  $C((2x/\pi)^{1/2})$  and  $S((2x/\pi)^{1/2})$ , in the preferred notation appearing in the FMRC *Index* [1], for example. The author here uses the unfortunate notation  $C(x)$  and  $S(x)$  for these forms of the Fresnel Integrals. The range of argument is  $x = 0(0.2)60$ , which exceeds somewhat that of the 6D table of Pearcey [2], which covers the range  $0(0.01)50$ .

Details of the computation of this table appear in a paper [3] published elsewhere in this journal.

J. W. W.

1. A. FLETCHER, J. C. P. MILLER, L. ROSENHEAD & L. J. COMRIE, *An Index of Mathematical Tables*, Vol. I, 2nd ed., Addison-Wesley Publishing Co., Reading, Mass., 1962, pp. 462-463.

2. T. PEARCEY, *Table of the Fresnel Integral to Six Decimal Places*, Cambridge Univ. Press, Cambridge, 1957. (See *MTAC*, v. 11, 1957, pp. 210-211, RMT 87.)

3. O. L. FLECKNER, "A method for the computation of the Fresnel integrals and related functions," *Math. Comp.*, v. 22, 1968, pp. 635-640.

67[7].—M. LAL, *Exact Values of Factorials 200! to 550!*, Department of Mathematics, Memorial University of Newfoundland, St. John's, Newfoundland, August 1967, ms. of iii + 152 pp., 28 cm., deposited in the UMT file.

68[7].—M. LAL & W. RUSSELL, *Exact Values of Factorials 500! to 1000!*, Department of Mathematics, Memorial University of Newfoundland, St. John's, Newfoundland, undated, ms. of ii + 501 pp., 28 cm., deposited in the UMT file.

The tabular contents of these companion manuscript volumes are clearly indicated by the respective titles. In the first table the factorials are printed *in extenso*; in the second, the terminal zeros are suppressed, but their number is recorded at the end of each entry. Furthermore, in the second table a separate page is allotted to each entry. In each table the digits are printed in five decades per line, with a space between successive arrays of ten lines. Also, the lines for each entry are consecutively numbered in the right margin.

The introduction to the first volume mentions the published table of Uhler [1] containing exact values of factorials to 200!, and also refers to subsequent related calculations [2], [3], [4] by that author. However, earlier, less extensive tabulations by others [5] are not cited.

The first table was computed at Dalhousie University by means of an IBM 1620 and an IBM 1132 printer; the second was computed at the Memorial University of Newfoundland by means of an IBM 1620 and an IBM 407 Mod E8 printer.

These impressive tables evolved as a by-product of a search for integer squares of the form  $n! + 1$  when  $n$  exceeds 7. This search, which has proved futile up to the limit  $n = 1140$ , extends earlier results of Kraitchik [6], as noted in the introduction to the first volume under review.

These attractive, clearly printed tables exemplify the excellent output obtainable from electronic digital computers in conjunction with meticulous planning and editing.

J. W. W.

1. H. S. UHLER, *Exact Values of the First 200 Factorials*, New Haven, 1944. (See *MTAC*, v. 1, 1943-1945, p. 312, RMT 158; p. 452, UMT 36.)
2. H. S. UHLER, "Twenty exact factorials between 304! and 401!", *Proc. Nat. Acad. Sci. U. S. A.*, v. 34, 1948, pp. 407-412. (See *MTAC*, v. 3, 1948-1949, p. 355, RMT 579.)
3. H. S. UHLER, "Nine exact factorials between 449! and 751!", *Scripta Math.*, v. 21, 1955, pp. 138-145.
4. H. S. UHLER, "Exact values of 996! and 1000!, with skeleton tables of antecedent constants," *Scripta Math.*, v. 21, 1955, pp. 261-268. (See *MTAC*, v. 11, 1957, p. 22, RMT 1.)
5. A. FLETCHER, J. C. P. MILLER, L. ROSENHEAD & L. J. COMRIE, *An Index of Mathematical Tables*, Vol. I, 2nd ed., Addison-Wesley Publishing Co., Reading, Mass., 1962, p. 47.
6. M. KRAITCHIK, "On the divisibility of factorials," *Scripta Math.*, v. 14, 1948, pp. 24-26. (See *MTAC*, v. 3, 1948-1949, pp. 357-358, RMT 587.)

**69[7].**—ALDEN McLELLAN IV, *Tables of the Riemann Zeta Function and Related Functions*, Desert Research Institute, University of Nevada, Reno, Nevada, ms. in 5 volumes, the first of 88 pp. the others of 83 pp. each, 28 cm., deposited in the UMT file.

In addition to the Riemann zeta function,  $\zeta(x)$ , here attractively tabulated to 41S (with respect to  $\zeta(x) - 1$ ) for  $x = 0(0.005)1(0.01)10(0.02)58$ , we find in the four accompanying tables decimal values of functions designated by the author as  $\alpha(x)$ ,  $\lambda(x)$ ,  $\eta(x)$ , and  $\xi(x)$ . The range here is  $x = 1(0.01)10(0.02)58$  and the precision is 41S, except for  $\lambda(x)$ , where from 31 to 40S of  $\lambda(x) - 1$  are tabulated. (All the tabular entries have been left unrounded.) These four functions can be expressed in terms of  $\zeta(x)$  by the relations:

$$\begin{aligned} \alpha(x) &= 2^{-x}\zeta(x), & \lambda(x) &= (1 - 2^{-x})\zeta(x), \\ \eta(x) &= (1 - 2^{-x+1})\zeta(x), & \xi(x) &= 2^{-x}(1 - 2^{-x+1})\zeta(x). \end{aligned}$$

Each of these functions has been previously tabulated; however, the earlier tables, except for those of  $\zeta(x)$ , have been restricted to integer values of the argument. Moreover, the notation employed in earlier tables, including those by Glaisher [1], Davis [2], and Liénard [3], differs from that adopted herein by Dr. McLellan. The two sets of notation are related as follows:

$$S_n = \zeta(n), \quad U_n = \lambda(n), \quad s_n = \eta(n), \quad 2^{-n}S_n = \alpha(n), \quad 2^{-n}s_n = \xi(n).$$

The present tables are not accompanied by any explanatory text; however, the introduction to a preliminary abridged table [4] by the same author reveals that the calculations were based upon Euler's transformation as applied to the alternating series derived from the standard series for  $\zeta(x)$  by means of van Wijngaarden's transformation [5]. Furthermore, this reviewer has ascertained that the calculations were performed on an IBM 1620 II computer, using a program written in machine language.

It might be noted that the most elaborate previous tabulation of  $\zeta(x)$  for decimal

arguments appears to have been made by Shafer [6], but his 30D manuscript table for  $x = 1.01(0.01)50$  is relatively inaccessible. For integer arguments the 50D tables of Liénard cover a wider range than those under review, but the precision is less for arguments exceeding 33.

Thus, the present manuscript tables, attractively arranged and clearly printed, represent a significant contribution to the tabular literature relating to the Riemann zeta function and associated functions.

J. W. W.

1. J. W. L. GLAISHER, "Tables of  $1 \pm 2^{-n} + 3^{-n} \pm 4^{-n} + \text{etc.}$  and  $1 + 3^{-n} + 5^{-n} + 7^{-n} + \text{etc.}$  to 32 places of decimals," *Quart. J. Math.*, v. 45, 1914, pp. 141-158.

2. H. T. DAVIS, *Tables of the Mathematical Functions*, Vol. II, Principia Press of Trinity University, San Antonio, Texas, 1963.

3. R. LIÉNARD, *Tables Fondamentales à 50 Décimales des Sommes  $S_n, U_n, \Sigma_n$* , Centre de Documentation Universitaire, Paris, 1948.

4. ALDEN McLELLAN IV, *Summing the Riemann Zeta Function*, Preprint No. 35, Desert Research Institute, University of Nevada, Reno, May 1966.

5. *Modern Computing Methods*, 2nd ed., Her Majesty's Stationery Office, London, 1961, p. 126.

6. A. FLETCHER, J. C. P. MILLER, L. ROSENHEAD & L. J. COMRIE, *An Index of Mathematical Tables*, Vol. I, 2nd ed., Addison-Wesley Publishing Co., Reading, Mass., 1962, p. 517.

70[7].—D. S. MITRINOVIĆ & R. S. MITRINOVIĆ, *Table des Nombres de Stirling de Seconde Espèce*, Publications de la Faculté d'Électrotechnique de l'Université à Belgrade (Série: Math. et Phys.), No. 181, 1967, 16 pp., 25 cm.

This attractive publication presents a table of the exact values of the Stirling numbers of the second kind, designated by  $\sigma_n^r$ , for  $r \leq n = 51(1)60$ .

The underlying calculations, performed on a desk calculator, were based on the recurrence relation  $\sigma_{n+1}^r = r\sigma_n^r + \sigma_n^{r-1}$ . Checking of the tabular entries corresponding to five selected values of  $n$  was performed at the Istituto Nazionale per le Applicazioni del Calcolo in Rome, using the relation  $\sum_{r=1}^n (r+1)\sigma_n^r = \sum_{r=1}^{n+1} \sigma_{n+1}^r$ .

In an addendum to the introduction the authors mention that this table was in the process of publication when they learned of the more extensive table by Andrew [1], with which they have found complete agreement.

The valuable list of references appended to the explanatory text includes the fundamental table of Gupta [2], which, as the authors explicitly note, has been inadvertently omitted as a reference in several earlier publications on these numbers.

J. W. W.

1. A. M. ANDREW, *Table of the Stirling Numbers of the Second Kind*, Tech. Rep. No. 6, Electrical Engineering Research Laboratory, Engineering Experiment Station, University of Illinois, Urbana, Illinois, December 1965. (See *Math. Comp.*, v. 21, 1967, pp. 117-118, RMT 3.)

2. H. GUPTA, "Tables of distributions," *Res. Bull. East Punjab Univ.*, No. 2, 1950, pp. 13-44. (See *MTAC*, v. 5, 1951, p. 71, RMT 859.)

71[7].—D. S. MITRINOVIĆ & R. S. MITRINOVIĆ, *Tableaux d'une classe de nombres reliés aux nombres de Stirling*, VII and VIII, Publ. Fac. Elect. Univ. Belgrade (Série: Math. et Phys.), Nos. 172 and 173, 1966, 53 pp., 24 cm.

The first part of the set of tables having the above title appeared in 1962; the seventh and eighth parts (forming a single fascicle) are stated to conclude this set. Reviews of all the earlier parts may be found in *Math. Comp.* (v. 17, 1963, p. 311,

RMT 44; v. 19, 1965, pp. 151–152, RMT 5, pp. 690–691, RMT 120; and v. 21, 1967, p. 264, RMT 24).

The seventh and eighth parts list exact values of  ${}^{\nu}S_n^k$  for  $n = 39$  and  $40$ , respectively; in each case for  $k = 1(1)n - 1$  and  $\nu = 1(1)n - 2$ . For  $n = 40$ ,  $k = 37$ ,  $\nu = 1$ , the tabular entry consists of 48 digits.

The integer  ${}^{\nu}S_n^k$  is defined as the coefficient of  $t^{n-k}$  in the product

$$t(t-1)(t-2)\cdots(t-\nu+1)(t-\nu-1)\cdots(t-n+1).$$

Exact values for  $n = 3(1)40$  are contained in this set of tables as a whole.

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72[8].—NORMAN DRAPER & HARRY SMITH, *Applied Regression Analysis*, John Wiley & Sons, New York, 1966, ix + 407 pp., 24 cm. Price \$11.75.

This text provides a standard, basic course in multiple linear regression. Topics of fundamental importance to regression analysis practitioners are included, beginning with fitting a straight line by least squares; then generalizing, by means of matrix notation, to multiple regression; and ending with a chapter devoted to nonlinear estimation.

Application of multiple regression to analysis of variance and covariance are considered. The emphasis is on practical applications. Many examples are included, and there are exercises at the end of nearly all the chapters, for which answers are provided. Examples of computer print-outs are also provided.

The book includes some material that is not generally available, for instance, a chapter on the examination of residuals. However, more consideration might have been given to the selection of an appropriate size sample and the related topic, power; the regression treatment of the two-way classification with an unequal number of observations in the cells (the nonorthogonal case); and canonical correlation (a generalization of multiple correlation). In the chapter on selecting the "best" regression equation, the Wherry "shrinkage" formula might have been considered. While it has some limitations, it seems more appropriate than the step-wise regression method for determining the number of predictors in the multiple-regression equation.

The text is authoritative and impressive. It should have an impact on the teaching of regression in universities. To readers who are familiar with multiple regression, it will serve as a very useful handbook.

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73[8, 13.35].—R. CRUON, Editor, *Queuing Theory: Recent Developments and Applications*, American Elsevier Publishing Co., New York, 1967, xv + 224 pp., 23 cm. Price \$13.50.

The title of the book suggests that the central theme of the conference in which these papers were presented was Queueing Theory and its Applications. However, a glance at the topics discussed discloses that the central theme has been stretched to include a wide variety of related topics.

The papers have been grouped into six sections, each containing papers presented at a particular session. These are: *Point Processes*, with papers "Processus Ponctuels dans les Modèles de Files d'Attente et de Stockage" by M. Girault, "A Self-Service System with Scheduled Arrivals" by A. Mercer and J. B. Parker, and "The Application of Stochastic Processes to Countdown Analysis" by David S. Stoller; *Mathematical Methods of Queueing Theory*, with papers "Pollaczek Method in Queueing Theory" by R. Syski, "Utilisation de la Théorie des Processus Semi-Markoviens dans l'Étude de Problèmes de Files d'Attente" by J. P. Lambotte and J. Teghem, and "The Custodian's Problem" by L. Kosten; *Nearly Saturated Queues and Transient Behavior*, with papers "Approximations for Queues in Heavy Traffic" by J. F. C. Kingman, "The Advantage of Precedence Operation with Overloaded or Fully Loaded Circuits" by E. P. G. Wright, and "Numerical Methods of Determining the Transient Behavior of Queues with Variable Arrival Rate" by E. L. Leese; *Inventories and Maintenance*, with papers "Files d'Attente et Stocks" by R. Debry and J. Teghem, and "A Mathematical Study of Unscheduled Open Hearth Furnace Repairs" by R. R. P. Jackson; *Statistical Problems*, with papers "Inference Statistique dans les Processus de Markov-Applications dans les Problèmes de Files d'Attente" by Simone Huyberechts, "Estimation of the Dispersion Parameter of an  $(A, B)$  Process" by L. J. Govier and T. Lewis, and "Spectral Analysis of Time Series Generated by Simulation Models" by G. S. Fishman and P. J. Kiviat; *Case Histories and Simulation* with papers "Checking the Simulation of a Queueing Type Situation" by B. W. Conolly, "A Study of a Multi-Channel Queueing Process in a Communications Terminal" by G. Hatzikostandis and S. Howe, "Arrivée à un bac d'un Traffic Composite" by B. Roy and J. Auberger, "Files d'Attente et Processus Markoviens à une Intersection de Traffic Comparison de Divers Modes de Contrôle" by P. Passau, and "Comparison Between Analogic and Digital Simulation of Telephone Gradings" by I. Capetti.

The book also includes a paper "The Application of Queueing Theory in Operations Research" by Philip Morse as Introduction, a paper "Ordering Disorderly Queues" by Thomas L. Saaty as Conclusion, and Final Remarks by R. Fortet.

Some of these papers have appeared before in one form or the other (for instance see E. L. Leese and D. W. Boyd "Numerical Methods of Determining the Transient Behavior of Queues with Variable Arrival Rates," *J. Canad. Operations Res. Soc.*, v. 4, 1966, pp. 1-13). In the reviewer's opinion the major contribution of the book seems to lie in exposing to the English speaking world some of the outstanding work being carried out in France and Belgium.

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74[9].—ALBERT H. BEILER, *Consecutive Hypotenuses of Pythagorean Triangles*, ms. of 11 typewritten pp. (including table) deposited in the UMT file.

The author concerns himself with sequences of  $r$  integers:

$$(1) \quad z, z + 1, z + 2, \dots, z + r - 1$$

such that each of these  $r$  numbers can be the hypotenuse of a Pythagorean triangle, while  $z - 1$  and  $z + r$  cannot so be. Thus, he has sequences of *exactly*  $r$  Pythagorean triangles which have consecutive hypotenuses. In his Table 1 he lists the first such  $z$  if it does not exceed  $10^5$ . There is such a  $z$  for  $r = 2(1)22$  and for  $r = 27$ , but for no other  $r$  except 1. In his table he gives the complete sequence (1) with each number completely factored and expressed as

$$(2) \quad z + i = Q(m^2 + n^2), \quad (n \neq m).$$

The number  $z + i$  can be written as (2) if and only if it has at least one prime divisor  $= 4k + 1$ , this being the criterion for  $z + i$  to be a Pythagorean hypotenuse.

We abbreviate his Table 1 by listing merely the first  $z < 10^5$  for each possible  $r$ . An industrious and interested reader could then reconstruct the sets of triangles.

$r$	$z$	$r$	$z$	$r$	$z$
2	25	3	39	4	50
5	218	6	775	7	949
8	673	9	403	10	1597
11	2190	12	2820	13	6050
14	8577	15	12423	16	27325
17	34075	18	52724	19	37088
20	74649	21	68150	22	43795
23	————	24	————	25	————
26	————	27	87594		

Thus, for  $r = 4$ , the triangles are

$$(14, 48, 50), (24, 45, 51), (20, 48, 52), (28, 45, 53).$$

For any  $r$ , the paper here shows how one could compute a set of *at least* (although not necessarily *exactly*)  $r$  triangles, but this method will usually not yield the smallest such  $z$ .

There is no discussion of the asymptotic behavior as  $z \rightarrow \infty$ . The whole tone of the paper could suggest to the reader that such long strings of  $r$  triangles are rare. Actually, as  $z \rightarrow \infty$  the opposite is true, since the numbers that *cannot* be written as (2) have zero density. This is clear, since such numbers can have as prime factors only 2 and the primes  $4k - 1$ . By a theorem due to Landau, cf. [1], the number of such (nonhypotenuse) numbers  $\leq x$  is asymptotic to

$$\frac{Ax}{\sqrt{\log x}}$$

for some computable constant  $A$ . While such numbers are common for small  $x$ , they take on a local density that (very gradually) goes to zero.

For the finite analogue of consecutive integers whose prime factors are only those taken from a given finite set, see Lehmer's analysis [2].

1. DANIEL SHANKS & LARRY P. SCHMID, "Variations on a theorem of Landau, Part I," *Math. Comp.*, v. 20, 1966, pp. 551-569. See also [2] of this paper.  
 2. D. H. LEHMER, "On a problem of Størmer,"<sup>1</sup> *Illinois J. Math.*, v. 8, 1964, pp. 57-79. Reviewed in RMT 67, *Math. Comp.*, v. 18, 1964, p. 510.

75[9].—MOHAN LAL & JAMES DAWE, *Solutions of the Diophantine Equation  $x^2 - Dy^4 = k$* , Memorial University of Newfoundland, October 1967, v + 122 pp. Deposited in the UMT file.

Bound in a gorgeous (cloudy-blue) loose-leaf binder are tables of solutions of

$$x^2 - Dy^4 = k$$

with  $D = 2(1)43$ , excluding the squares:  $D = 4, 9, 16, 25, 36$ , with  $|k| \leq 999$ , and with  $y \leq 200,000$ . All solutions for  $D = 2$  are listed first, with  $|k|$  in ascending order, then those with  $D = 3$ , etc. The *imprimitive* solutions are those with

$$x^2 = a^2b^2, \quad D = a^2d, \quad k = a^2K,$$

or with

$$x^2 = z^2a^4, \quad y^4 = a^4b^4, \quad k = a^4K.$$

These are marked with an asterisk. There are 2672 primitive solutions with  $k$  positive and 2150 primitive solutions with  $k$  negative. The tables are prefaced by their note [1] appearing elsewhere in this issue.

Here are some observations obtained by casual examination of the tables.

A. Although solutions were sought with  $y \leq 200,000$ , there are, in fact, none here with  $y > 6227$ . This "largest" solution is

$$54836879^2 - 2 \cdot 6227^4 = 959.$$

This strengthens, some, the Result #1 in [1], and suggests that for most pairs  $[D, k]$ , at least, the sets of solutions  $(x, y)$  here are *complete*. The authors cautiously refrain from drawing this inference.

B. The maximum number (six) of primitive solutions, occurs for  $[D, k] = [3, 913]$  and  $[19, 657]$ . Specifically, for the first,  $(x, y) = (31, 2), (34, 3), (41, 4), (626, 19), (51241, 172)$ , and  $(1292969, 864)$ , while for the second we have  $(26, 1), (31, 2), (159, 6), (354, 9), (2306, 23)$ , and  $(53706, 111)$ .

C. Whenever  $D$  is not of the form  $u^2 + v^2$ , we cannot have solutions for both  $[D, k]$  and  $[D, -k]$ , since that would imply the impossible equation

$$x_1^2 + x_2^2 = D(y_1^4 + y_2^4).$$

Thus, from the two cases above, there are *no* solutions for  $[3, -913]$  or  $[19, -657]$ . But, *more generally*, it is noted that whenever  $[D, k]$  has six, five, or four primitive solutions, then  $[D, -k]$  has no solution here, whether or not  $D = u^2 + v^2$ . This is very similar to our observation [2] concerning

$$y^3 - x^2 = \pm k.$$

For some cases of three solutions, such as  $[D, k] = [5, -44]$ , or  $[5, -209]$ , one does find a solution for  $[D, -k]$ . No such three-and-one set is found, however, if  $D = 2$ .

D. A curiosity, following from the last point, is that [2, 959] has five solutions, [2, -959] none, and [8, -959] five; [2, -161] four solutions, [2, 161] none, and [8, 161] four. While one naturally suspects some algebraic relationship here, none was discovered by the reviewer—probably through insufficient diligence. The result is not general. Thus, [2, 194] has three solutions while both [8, 194] and [8, -194] have none.

D. S.

1. MOHAN LAL & JAMES DAWE, "Solutions of the Diophantine equation  $x^2 - Dy^4 = k$ ," *Math. Comp.*, v. 22, 1968, pp. 679-682.

2. RMT 89, *Math. Comp.*, v. 20, 1966, pp. 620-621.

76[10].—SAUNDERS MACLANE & GARRETT BIRKHOFF, *Algebra*, The Macmillan Co., New York, 1967, xix + 598 pp., 24 cm. Price \$11.95.

In spite of the similarity of the titles and the coincidence of the names (although not the sequence of names) of the authors, this is not a new edition of the *Survey of Modern Algebra* (Macmillan Co., 1953) but a new book.

The motivation for it is summarized in the first paragraph of the Preface: "Recent years have seen striking developments in the conceptual organization of mathematics. These developments use certain new concepts such as 'module,' 'category,' and 'morphism' which are algebraic in character and which indeed can be introduced naturally on the basis of elementary materials. The efficiency of these ideas suggests a fresh presentation of algebra."

As in the *Survey of Modern Algebra*, the concepts and basic facts of the theory of sets, integers, groups, rings, fields, matrices, and vector spaces are introduced and proved; in addition, modules, lattices, multilinear algebra and other topics have their own chapters and are treated either in greater detail or as new subjects. But most of these chapters are used now also for the purpose of introducing and illustrating the concepts of "category," "functor," and "universal element" which, in the penultimate chapter on *Categories and adjoint functors* become the main topic of the book. Functors on sets to sets are introduced on page 24. The definition of a universal element appears on page 26. Its description as "the most important concept in algebra" seems to refer to the present book rather than to algebra as a discipline (but this is indicated merely by the italicizing of the word "algebra").

Concrete categories are introduced on page 64. However, most of the theorems and proofs in the book can be read without knowledge of the theory of categories.

The book is extremely well organized and very well written. Examples illustrating a new concept are given immediately after its definition. Chapters and even sections are preceded by brief, summarizing statements. Theorems are followed frequently by elucidating comments. Proofs are chosen on the basis of transparency rather than brevity; for instance, the first Sylow theorem is proved in the traditional manner without using Wielandt's elegant combinatorial argument, and the Jordan-Hoelder theorem is proved without using the powerful but difficult lemma of Zassenhaus.

The omission of Galois Theory (which had a brief but important chapter in the "Survey") is deplored by the authors and is easily explained by the fact that the present book has 598 pages versus 472 of the "Survey" which also had a smaller format. However, this very fact indicates a serious difficulty arising in the teaching

of mathematics (or at least of algebra). Galois died 136 years ago. His theory of algebraic equations (or finite extensions of fields) is still a highly relevant and important part of algebra. But to get acquainted with it seems to require an increasing amount of studies. We know (Genesis 29, 30) that Jacob served for Rachel not only seven years but yet seven other years. Will it be the fate of our students to reach the goal of their studies only at the age granted to patriarchs?

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