Numerical Methods and Existence Theorems for Parabolic Differential Equations whose Coefficients are Singular on the Boundary

By Pierre Jamet

I. Introduction. In a previous paper [6], S. V. Parter and the author have studied finite-difference methods for elliptic differential equations of the second order whose coefficients are singular on a portion of the boundary; the uniform convergence of the approximations and the existence of a solution of the Dirichlet problem were proved for a class of such equations. The present work is an extension of those results to parabolic initial boundary-value problems. The class of problems that we consider includes the cases of nonhomogeneous differential equations, of time-dependent coefficients, of time-dependent domains and of over-determined Dirichlet problems.

Let G be a bounded (open) domain in R^n and let $P = (x_1, \cdots, x_n)$ denote an element of G. Let L be a differential operator of the form

$$(1.1) \qquad Lu = \sum_{r,s=1}^{n} a_{rs} \frac{\partial^2 u}{\partial x_r \partial x_s} + \sum_{r=1}^{n} b_r \frac{\partial u}{\partial x_r} - cu .$$

The coefficients $a_{rs} = a_{sr}$, b_r and c are functions of P; we assume that they are "smooth"* in the interior of G; but they may be singular, for instance be unbounded, as P approaches the boundary ∂G of G. Moreover, we assume

$$(1.2) \qquad \sum_{r,s=1}^{n} a_{rs}(P)\xi_r\xi_s \geqq 0 , \qquad \forall (\xi_1, \cdots, \xi_n) \neq 0, \forall P \in G ,$$

$$(1.3) \qquad c(P) \geqq 0 , \qquad \forall P \in G .$$

The work in [6] was devoted to the elliptic case:

$$(1.4) \qquad \sum_{r,s=1}^{n} a_{rs}(P)\xi_r\xi_s > 0 , \qquad \forall \{\xi_1, \cdots, \xi_n\} \neq 0, \forall P \in G .$$

In the present paper, we are primarily interested in the parabolic case:

$$(1.5) \qquad \begin{aligned} &a_{rn}(P) = 0 , \qquad r = 1, 2, \cdots, n , \\ &\sum_{r,s=1}^{n-1} a_{rs}(P)\xi_r\xi_s > 0 , \qquad \forall \{\xi_1, \cdots, \xi_{n-1}\} \neq 0 , \\ &b_n(P) < 0 , \qquad \forall P \in G . \end{aligned}$$

In this case we shall write $x_n = t$ (time variable). However, for greater generality, we will take, at first, the operator L in the form (1.1) and we will only assume conditions (1.2) (1.3).

Let Γ_1 and Γ_2 be two complementary subsets of ∂G; $\Gamma_1 \neq \varnothing$. Let $f(P)$ be a

Received December 4, 1967.

* We need not specify now the degree of smoothness.

bounded function defined on $\bar{G}$ and which is "smooth" in the interior of G; let $g(P) \in C(\bar{G})$. We consider the differential equation

$$Lu = f \tag{1.6}$$

and the boundary value problem

$$\begin{aligned} Lu(P) &= f(P)\,, && P \in G\,, \\ u(P) &= g(P)\,, && P \in \Gamma_1\,, \\ u(P) &\in C^2(G) \cap C(G \cup \Gamma_1) \cap B(G)\,, \end{aligned} \tag{1.7}$$

where $B(G)$ denotes the space of all bounded functions on G.

We say that (1.7) is a problem of "Dirichlet type"*. Of course, Γ_1 cannot be chosen arbitrarily if we want problem (1.7) to admit a unique solution; this choice depends on the type of the operator and on the singularities of its coefficients near the boundary. A simple example is the following:

Let $G \subset R^2$ be the triangle $0 < x < t < 2$ and let

$$Lu \equiv \frac{\partial^2 u}{\partial x^2} + \frac{t}{x}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}\,.$$

Suppose $f(P) \in C^3(G) \cap B(G)$. Then, problem (1.7) has a unique solution provided we take:

$$\Gamma_1 = \{P = (x, t); 0 \leqq t = x \leqq 2\} \cup \{P = (0, t); 0 \leqq t < 1\}\,.$$

Problems of the type (1.7) have been studied by J. J. Kohn and L. Nirenberg [7]; these authors give results concerning existence, unicity and regularity of the solution; however, our hypotheses are different from theirs and, therefore, our existence and unicity theorems are also different. Finite-difference schemes for time-dependent problems with singular coefficients have been studied by D. Eisen [4]; this author studies the relations between stability and convergence, in the framework of the Lax-Richtmyer theory [11].

In Section 2 of the present paper, we recall the basic convergence and existence argument which was used in [6]; it is based on the notion of "discrete barrier"; the presentation is more general than in [6], which is necessary for the applications to a wider class of problems. Our fundamental Theorem 2.1 reduces the questions of convergence and existence to three independent questions which are studied in the three following sections: uniform boundedness of the approximations, interior equicontinuity and existence of local discrete barriers. Section 6 is devoted to the problem of unicity. Finally, Section 7 is an account of numerical experiments.

II. Finite-Difference Schemes and Discrete Barriers.

1. *Generalities.* Let h be a parameter (for instance an n-vector with positive components) and let $\bar{G}(h)$ be for each h a finite set of points in $\bar{G}$ with the following property:

$$\sup_{P \in G} d(P, \bar{G}(h)) \to 0 \quad \text{as } h \to 0\,.^{**} \tag{2.1}$$

* It is of no significance for this problem to know the values of $f(P)$ on ∂G or the values of $g(P)$ on $G \cup \Gamma_2$. But we will need those values for the discrete analogue of this problem; they can be chosen arbitrarily.

** We denote by $d(E, E')$ the distance between two sets E and E' in R^n.

Let $G(h)$ and $\partial G(h)$ be two complementary nonempty subsets of $\overline{G}(h)$. We assume that

$$\max_{P \in \partial G(h)} d(P, \partial G) \to 0 \quad \text{as } h \to 0 .^* \tag{2.2}$$

To each point $P \in G(h)$ we associate a set $\mathfrak{N}(P) \subset \overline{G}(h) - \{P\}$ which is called the set of neighbor points of P in $\overline{G}(h)$ and which satisfies

$$\max_{P \in G(h)} \max_{P' \in \mathfrak{N}(P)} d(P, P') \to 0 \quad \text{as } h \to 0 . \tag{2.3}$$

We assume that, for h small, $\overline{G}(h)$ has the following "connectedness" property: $\forall P \in G(h)$, $\exists$ a sequence of points $P_0, P_1, \cdots, P_r$ such that

$$\begin{aligned} &P_0 = P , \\ &P_1, P_2, \cdots, P_{r-1} \in G(h) , \\ &P_r \in \partial G(h) , \\ &P_{i+1} \in \mathfrak{N}(P_i) , \qquad i = 0, 1, \cdots, (r-1) . \end{aligned} \tag{2.4}$$

Let $v(P)$ be a function defined on $\overline{G}(h)$. At each point $P \in G(h)$, we define

$$L_h v(P) = -A(P, P)v(P) + \sum_{P' \in \mathfrak{N}(P)} A(P, P')v(P') . \tag{2.5}$$

We assume that, for h small, the operator L_h is of positive type, i.e., for all $P \in G(h)$

$$\begin{aligned} A(P, \ P) > 0; \qquad A(P, P') > 0, \qquad \forall P' \in \mathfrak{N}(P) , \\ E(P) \equiv A(P, P) - \sum_{P' \in \mathfrak{N}(P)} A(P, P') \geqq 0 . \end{aligned} \tag{2.6}$$

Under such hypotheses, the following maximum principle holds: let $v(P)$ be any function defined on $\overline{G}(h)$ and such that $L_h v(P) \geqq 0$, $\forall\, P \in G(h)$; then

$$\max_{P \in G(h)} v(P) \leqq \max \left\{0, \max_{P \in \partial G(h)} v(P)\right\} .$$

Now, we introduce some notations and definitions which will be used later. Given any subdomain G' of G, we define

$$\begin{aligned} \overline{G}'(h) &= \{P \in \overline{G}(h) \cap \overline{G}'\} , \\ G'(h) &= \{P \in G(h) \cap \overline{G}'; \mathfrak{N}(P) \in \overline{G}'\} , \\ \partial G'(h) &= \overline{G}'(h) - G'(h) . \end{aligned} \tag{2.7}$$

Definition 2.1. *Uniform consistency*. Let $G' \subset G$. We say that L_h is a uniformly consistent approximation to the operator L in G' if, $\forall \phi \in C^2(\overline{G}')$,

$$\max_{P \in G'(h)} |L_h \phi(P) - L\phi(P)| \to 0 \quad \text{as } h \to 0 .$$

Definition 2.2. *Discrete equicontinuity*. Let $G' \subset G$ and $\mathfrak{F} = \{v(P; h)\}$ be a family of mesh-functions defined on $\overline{G}(h)$ for each h. We say that the family $\mathfrak{F}$ is equicontinuous in G' if, given any $\epsilon > 0$, there exists a constant $\eta > 0$ independent of h such that $|v(P; h) - v(P'; h)| < \epsilon$, $\forall P, P' \in \overline{G}'(h)$ such that $d(P, P') < \eta$.

Definition 2.3. *Discrete uniform convergence*. Let $G' \subset G$. Let $\{v(P; h)\}$ be a

* It is important to observe that we do *not* assume $G(h) \cap \partial G = \emptyset$.

family of mesh-functions defined on $\overline{G}(h)$ for each h and let $u(P)$ be a function defined on $\overline{G}'$. We say that $v(P; h)$ converges uniformly to $u(P)$ in G' as $h \to 0$ if

$$\operatorname*{Max}_{P \in \overline{G}'(h)} |v(P; h) - u(P)| \to 0 \quad \text{as } h \to 0 .$$

Now, let us consider an infinite family $\{h\}$ of parameters h, with zero as an adherence point, and the corresponding family $\{L_h\}$ of operators.

Definition 2.4. *Discrete barrier.* Let $Q \in \partial G$. A function $B(P; Q)$ is a strong (local) discrete barrier at the point Q relative to the family $\{L_h\}$ if there exists a neighborhood N of the point Q in the relative topology of $\overline{G}$ such that

$$B(P; Q) \in C(N) , \tag{2.8a}$$

$$B(Q; Q) = 0 , \tag{2.8b}$$

$$B(P; Q) < 0 , \qquad \forall P \in N - \{Q\} , \tag{2.8c}$$

$$L_h B(P; Q) - E(P) \geqq 1 , \qquad \forall P \in N(h) \text{ and } \forall h \text{ small enough.} \tag{2.8d}$$

Now we consider the following system of linear equations

$$\begin{aligned} L_h v(P) &= f(P) , \quad P \in G(h) , \\ v(P) &= g(P) , \quad P \in \partial G(h) . \end{aligned} \tag{2.9}$$

It follows from our hypotheses that, for h small enough, this system has a unique solution $v(P; h)$; this is a direct consequence of the maximum principle.

THEOREM 2.1. *Let* $\mathfrak{F} = \{v(P; h)\}$ *be the family of the solutions of* (2.9) *for all* h *small enough. Let us assume*

(i) *There exists a function* $\phi(P) \in C(\overline{G})$ *such that* $L_h\phi(P) \geqq 1$, $\forall P \in G(h)$ *and for all* h.

(ii) *For any* $G' \subset\subset G$ † *and for any sequence* $\{v(P; h_n); h_n \to 0\} \subset \mathfrak{F}$, *there exists a subsequence which converges uniformly in* G' *to a solution of Eq.* (1.6).

(iii) *At each point* $Q \in \Gamma_1$, *there exists a strong* (*local*) *discrete barrier relative to the family* $\{L_h\}$.

Then, problem (1.7) *has at least one solution* $u(P)$. *Moreover, if this solution is unique,* $v(P; h)$ *converges to* $u(P)$ *as* $h \to 0$, *uniformly in* $\overline{G} - N(\Gamma_2)$ *where* $N(\Gamma_2)$ *is an arbitrary neighborhood of* Γ_2.

Proof. The proof of this theorem is a modification of the proof of Theorem 2.3 in [6]. We shall concentrate mostly on those modifications and refer the reader to [6] for more details.

We observe that assumption (i) implies the uniform boundedness in $\overline{G}$ of the family $\{v(P; h)\}$; this follows from the maximum principle and from the boundedness of $f(P)$; we denote by M a uniform bound for $|v(P; h)|$. Let $Q \in \Gamma_1$ and let $B(P; Q)$ be a strong discrete barrier at Q; let N be a neighborhood of Q for which conditions (2.8) are satisfied; we can write $N = N_0 \cap \overline{G}$ where N_0 is a neighborhood of Q in R^n. Let $N_0' \subset\subset N_0$ be also a neighborhood of Q in R^n and let $N' = N_0' \cap \overline{G}$. It follows from assumption (2.3) and definition (2.7) (applied to the subdomain N) that, for h small enough:

$$\partial N(h) \subset \partial G(h) \cup (N - N') . \tag{2.10}$$

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† $G' \subset\subset G$ is an abreviation for $G' \subset \overline{G}' \subset G$. We say that G' is an "interior" subset of G.

Let $M' = \mathrm{Sup}_{P \in N} |f(P)|$; let $\epsilon > 0$ be arbitrary and let us consider the two functions

$$F(P) = g(Q) - \epsilon + \eta B(P; Q) , \quad G(P) = g(Q) + \epsilon - \eta B(P; Q) , \tag{2.11}$$

where η is so large that

$$\eta > \mathrm{Max}\,\{M', |g(Q)|\} , \quad F(P) \leqq g(P) \leqq G(P) , \quad \forall P \in N , \quad F(P) \leqq -M < M \leqq G(P) , \quad \forall P \in N - N' . \tag{2.12}$$

It is easy to check that, for h small enough:

$$L_h F(P) \geqq L_h v(P; h) \geqq L_h G(P) , \qquad \forall P \in N(h) ,$$
$$F(P) \leqq v(P; h) \leqq G(P) , \qquad \forall P \in \partial N(h) .$$

Therefore, using the maximum principle, we get

$$F(P) \leqq v(P; h) \leqq G(P) , \qquad \forall P \in \overline{N}(h) .$$

The rest of the argument is the same as in [6].

Remark. Theorem 2.1 holds, more generally, for all monotone finite-difference operators such that $E(P) \geqq 0$, $\forall P \in G(h)$.

2. *The parabolic case.* In the following sections we will restrict our attention to the parabolic case (1.5); moreover we will assume $a_{rs} \equiv 0$ if $r \neq s$. We assume that the coefficients of the equation and the function $f(P)$ are in $C^\infty(G)$. All of what follows is valid for any n, but, to avoid complications in the notations, we will assume $n = 3$ and we will write: $x_1 = x$, $x_2 = y$, $x_3 = t$ and

$$Lu = a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial x} + a' \frac{\partial^2 u}{\partial y^2} + b' \frac{\partial u}{\partial y} - cu - d \frac{\partial u}{\partial t} , \tag{2.13}$$

where $a(P)$, $a'(P)$, $d(P) > 0$ and $c(P) \geqq 0$ for all $P \in G$.

Let h be a positive number and let us consider the square net

$$R(h) = \{P = (ih, jh, kh);\, i, j, k \text{ integers}\}.^*$$

To any point $P = (ih, jh, kh) \in R(h)$ we associate a set $\mathfrak{N}_0(P)$, which consists of the five points

$$((i \pm 1)h, jh, kh) , \qquad (ih, (j \pm 1)h, kh) , \qquad (ih, jh, (k - 1)h) .$$

Let $\overline{\mathfrak{N}}_0(P)$ be the set of the five segments joining the point P to each of the points of $\mathfrak{N}_0(P)$. We define

$$\overline{G}(h) = \overline{G} \cap R(h) ,$$
$$G_0(h) = \{P \in \overline{G}(h); \overline{\mathfrak{N}}_0(P) \subset \overline{G}\} ,$$
$$\Gamma_1(h) = \{P \in \overline{G}(h) - G_0(h); d(P, \Gamma_1) < h\} .$$

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* For greater simplicity we consider a square net instead of a rectangular mesh; of course, this is not essential.

We choose $\Gamma(h)$ and $G(h)$ arbitrarily provided $\Gamma_1(h) \subset \Gamma(h)$ and $G_0(h) \subset G(h)$. Now, we must define the set $\mathfrak{N}(P)$ and the operator L_h at each point $P \in G(h)$. We do this in the following way: at each point $P \in G(h) - G_0(h)$ we define $\mathfrak{N}(P)$ arbitrarily provided $\mathfrak{N}(P) \cap G_0(h) \neq \phi$, and at each point $P \in G_0(h)$ we take $\mathfrak{N}(P) = \mathfrak{N}_0(P)$; this choice guarantees the "connectedness" of $\overline{G}(h)$ for h small. At each point $P \in G(h) - G_0(h)$ we define L_h arbitrarily provided conditions (2.6) are satisfied at that point. At each point $P \in G_0(h)$ the choice of L_h depends on the operator L; let $v_x, v_{\bar{x}}, v_y, \cdots$ denote the usual forward and backward difference quotients of the function v; we define

$$L_h v(P) = \alpha v_{x\bar{x}} + \beta \frac{v_x + v_{\bar{x}}}{2} + \alpha' v_{y\bar{y}} + \beta' \frac{v_y + v_{\bar{y}}}{2} - \gamma v - \delta v_{\bar{t}} \,, \tag{2.14}$$

where the coefficients $\alpha, \beta, \cdots, \delta$ are functions of P and h, and are related to the coefficients of the operator L; here are two possible choices for those coefficients:

First choice.

$$\begin{aligned} \alpha(P; h) &= a(P) \,, \\ \beta(P; h) &= b(P) \,, \\ &\cdots\cdots \\ \delta(P; h) &= d(P) \,. \end{aligned} \tag{2.15}$$

Second choice. For each $P = (x, y, t) \in G_0(h)$, let

$$\alpha_+(P; h) = \exp \int_x^{x+h/2} \frac{b(z, y, t)}{a(z, y, t)}\, dz \,,$$

$$\alpha_-(P; h) = \exp \int_x^{x-h/2} \frac{b(z, y, t)}{a(z, y, t)}\, dz \,,$$

$$\alpha_+'(P; h) = \exp \int_y^{y+h/2} \frac{b'(x, z, t)}{a'(x, z, t)}\, dz \,,$$

$$\alpha_-'(P; h) = \exp \int_y^{y-h/2} \frac{b'(x, z, t)}{a'(x, z, t)}\, dz \,,$$

and

$$\begin{aligned} \alpha(P; h) &= a(P) \frac{\alpha_+(P; h) + \alpha_-(P; h)}{2} \,, \\ \beta(P; h) &= a(P) \frac{\alpha_+(P; h) - \alpha_-(P; h)}{h} \,, \\ \alpha'(P; h) &= a'(P) \frac{\alpha_+'(P; h) + \alpha_-'(P; h)}{2} \,, \\ \beta'(P; h) &= a'(P) \frac{\alpha_+'(P; h) - \alpha_-'(P; h)}{h} \,, \\ \gamma(P; h) &= c(P) \,, \\ \delta(P; h) &= d(P) \,. \end{aligned} \tag{2.16}$$

Of course, the choice (2.15) is more natural; it is also easier, except in particular cases for which the integrals above admit simple analytic representations. However, this choice is not always suitable, because we want the operator L_h to be of positive type for h small; this condition is not always satisfied for the operator corresponding to the choice (2.15): it depends on the behavior of the coefficients of the operator L near the boundary. On the contrary, the operator corresponding to the choice (2.16) is always of positive type; to see this, we observe that at each point $P \in G_0(h)$ this operator can be written in the form

$$L_h v(P) = a \frac{\alpha_+ v_x - \alpha_- v_{\bar{x}}}{h} + a' \frac{\alpha_+' v_y - \alpha_-' v_{\bar{y}}}{h} - \gamma v - \delta v_{\bar{t}} \,, \tag{2.17}$$

where all the coefficients a, α_+, α_-, $\cdots$, δ are nonnegative.

We will need also the two following properties of the operator L_h, which are satisfied for both choices (2.15) and (2.16):

(A) The operator L_h is a uniformly consistent approximation to L in any interior subdomain $G' \subset\subset G$. This follows from the relations

$$\begin{aligned} \alpha(P;h) &= a(P) + O(1) \,, \\ \beta(P;h) &= b(P) + O(1) \,, \\ &\cdots\cdots\cdots \\ \delta(P;h) &= d(P) + O(1) \,, \end{aligned} \tag{2.18}$$

which hold uniformly in G' for h small.*

(B) Given any interior subdomain $G' \subset\subset G$ and any positive integer p, all the difference quotients of order p of the coefficients α, β, $\cdots$, δ are uniformly bounded for all $P \in G'(h)$ and for all h sufficiently small.

III. Uniform Boundedness in the Nonhomogeneous Case. In order to apply Theorem 2.1 to inhomogeneous problems, it is necessary to study the existence of a function $\phi(P)$ which satisfies condition (i). The existence of such a function guarantees the uniform boundedness of the approximations $v(P; h)$. We give here a few simple criterions for the existence of $\phi(P)$.

Let L be the operator (2.13). Let $G(h) = G_0(h)$ and let L_h be defined by formula (2.14) together with (2.15) or (2.16).

1. *First sufficient condition.* Suppose $c(P) > m > 0$ in G, then there exists a function $\phi(P)$ which satisfies condition (i) of Theorem 2.1.

Proof. Take $\phi(P) = -1/m$.

2. *Second sufficient condition.* Suppose $d(P) > m > 0$ in G. Same conclusion.

Proof. Take $\phi(P) = -(K + t/m)$ where $K > 0$ is chosen so large that $\phi(P) < 0$ in G.

3. *Third sufficient condition.* Suppose $a(P) > m > 0$ and $|b(P)| < M$ in G. Same conclusion.

Proof. Take $\phi(P) = K[\exp(\rho x) - K']$, with $\rho > M/m$ and K, K' sufficiently large.

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* It is interesting to note that conditions (2.18) are also *necessary* for the uniform consistency of the operator L_h to the operator L in G'.

IV. Interior Estimates. Let L_h be a finite-difference operator of positive type which has the form (2.14) for all $P \in G_0(h)$ and which satisfies properties (A) and (B) (see the end of Section II). Let $\mathfrak{F} = \{v(P; h)\}$ be a family of mesh-functions defined on $\overline{G}(h)$ for each h and such that $L_h v(P; h) = f(P), \ \forall P \in G_0(h)$. Let $\mathfrak{F}^{(p)}$ be the family of all difference quotients of order p of the functions of $\mathfrak{F}$. Let G' be an arbitrary interior subdomain of G.

THEOREM 4.1. *If the family $\mathfrak{F}$ is uniformly bounded in G, then it is equicontinuous in G'. Moreover, each family $\mathfrak{F}^{(p)}$ is equicontinuous in G'.*

This theorem is an extension to parabolic problems of a well-known theorem for elliptic problems, which is due to Courant, Friedrichs and Lewy [1]. These authors proved this theorem in the particular case of the Laplacian operator in two dimensions; more general proofs were given later by W. V. Koppenfels [8] for general elliptic operators in two dimensions and by C. Cryer [3] for elliptic operators in R^n. Those proofs are based on a discrete analogue of Sobolev's imbedding theorem (see [12]) which was first discovered by Courant, Friedrichs and Lewy in the case $n = 2$: let $G' \subset\subset G$ and let $\mu > n/2$ be an integer; assume that the sums $h^n \sum_{G'(h)} w^2(P; h)$ are uniformly bounded for all $w(P; h)$ which are difference quotients of order $\leqq \mu$ of the functions of $\mathfrak{F}$; then the family $\mathfrak{F}$ is equicontinuous in any subdomain $G'' \subset\subset G'$.

This theorem shows that we have only to prove the uniform boundedness of the sums $h^n \sum_{G'(h)} w^2(P; h)$. This proof is based on the discrete analogue of Green's formula (see Cryer [2]). To avoid complications, we will develop the argument only in the case $n = 2$, i.e., we consider only two independent variables x and t; it is clear that this argument which is only a modification of the argument used by Courant, Friedrichs and Lewy in the elliptic case, can be extended to R^n in the same way as in the elliptic case.

Let h be so small that $G'(h) \subset G_0(h)$. Then, at each point $P \in G'(h)$, we have

$$L_h v \equiv \alpha v_{x\bar{x}} + \beta(v_x + v_{\bar{x}})/2 - \gamma v - \delta v_{\bar{t}} = f , \tag{4.1}$$

where the coefficients α, β, γ and δ satisfy conditions (2.18) and $\gamma \geqq 0$. We shall assume $\delta = 1$, which is not a restriction. For h small enough, we have

$$\begin{gathered} 0 < m < \alpha(P; h) < M , \\ |\beta(P; h)|, |\gamma(P; h)|, |f(P; h)| < M , \end{gathered} \tag{4.2}$$

for all $P \in G'(h)$ and for some suitable constants m and M. We will assume that M is also an upper bound for $\mathfrak{F}$ and for any of the difference quotients of α, β, γ and f which will be used in the proof. It will be convenient to write $L_h v \equiv L_h^0 v - v_{\bar{t}}$ where L_h^0 denotes the space-operator

$$L_h^0 v \equiv \alpha v_{x\bar{x}} + \beta(v_x + v_{\bar{x}})/2 - \gamma v .$$

Let h be fixed (sufficiently small so that the preceding conditions are satisfied); following Courant, Friedrichs and Lewy, we consider an expanding sequence of concentric rectangles in $G'(h)$, say $\{Q_0, Q_1, \cdots, Q_k, \cdots, Q_N\}$, such that

$$Q_k = \{P = (x, t) = (ih, jh) \in G'(h); i_k \leqq i \leqq i_k', j_k \leqq j \leqq j_k'\} ,$$

$$i_{k+1} = i_k - 1 , \quad i'_{k+1} = i_k' + 1 , \quad j_{k+1} = j_k - 1 , \quad j'_{k+1} = j_k' + 1 .$$

We define $S_k = Q_k - Q_{k-1}$ and $R_k = \{P = (ih, jh) \in S_k; i = i_k \text{ or } i_k'\}$.

First we prove the following lemma:

LEMMA 4.1. *For every function $w(P)$ defined on $\overline{G}(h)$, the following inequality holds*

$$
\begin{aligned}
\left(m - \frac{\kappa}{2} M\right) h^2 \sum_{Q_{k-1}}\sum (w_x{}^2 + w_{\bar{x}}{}^2) &\leqq 2h^2 \sum_{Q_{k-1}}\sum |w||L_h w| \\
&+ \left(\sum_{R_k} \alpha w^2 - \sum_{R_{k-1}} \alpha w^2\right) + M\left(1 + \frac{1}{\kappa}\right) h^2 \sum_{Q_k}\sum w^2 \\
&+ h\left(\frac{M}{2} + 1\right)\left(\sum_{S_k} w^2 + \sum_{S_{k-1}} w^2\right),
\end{aligned} \tag{4.3}
$$

where κ is any positive number and $1 \leqq k \leqq N$.

Proof. We will make the following convention: for any function $w(P)$ defined on $\overline{G}(h)$, we denote

$$
\begin{aligned}
w &= w_{ij} = w(ih, jh)\ , \\
w_{i+1} &= w_{i+1,j} = w((i+1)h, jh)\ , \\
w_{j+1} &= w_{i,j+1} = w(ih, (j+1)h, \cdots)\ ,
\end{aligned}
$$

i.e., we drop the first index each time it has the value i and the second one each time it has the value j.

Using those notations, we define

$$
\begin{aligned}
A(w) &= \alpha_{i+1} w_x{}^2 + \alpha_x w w_x + (w/2)(\beta w)_x - (\beta/2) w w_x + \gamma w^2\ , \\
\overline{A}(w) &= \alpha_{i-1} w_{\bar{x}}{}^2 + \alpha_{\bar{x}} w w_{\bar{x}} + (w/2)(\beta w)_{\bar{x}} - (\beta/2) w w_{\bar{x}} + \gamma w^2\ .
\end{aligned}
$$

Let $j_0 \leqq j \leqq j_0'$. An elementary manipulation based on summation by parts gives

$$
\begin{aligned}
h^2 \sum_{i=i_0}^{i_0'} (A(w) + \overline{A}(w)) &= -2h^2 \sum_{i=i_0}^{i_0'} w L_h{}^0 w \\
&\quad + [\alpha_{i_1} w_{i_1}^2 - \alpha_{i_0} w_{i_0}^2 + (h/2)(\beta_{i_1} + \beta_{i_0}) w_{i_0} w_{i_1}] \\
&\quad + [\alpha_{i_1'} w_{i_1'}^2 - \alpha_{i_0'} w_{i_0'}^2 - (h/2)(\beta_{i_1'} + \beta_{i_0'}) w_{i_0'} w_{i_1'}] \\
&\leqq -2h^2 \sum_{i=i_0}^{i_0'} w L_h{}^0 w + (\alpha_{i_1} w_{i_1}^2 - \alpha_{i_0} w_{i_0}^2) \\
&\quad + (\alpha_{i_1'} w_{i_1'}^2 - \alpha_{i_0'} w_{i_0'}^2) \\
&\quad + h(M/2)(w_{i_0}^2 + w_{i_1}^2 + w_{i_0'}^2 + w_{i_1'}^2)\ .
\end{aligned}
$$

Summing from $j = j_0$ to $j = j_0'$, we get

$$
\begin{aligned}
h^2 \sum_{Q_0}\sum (A(w) + \overline{A}(w)) &\leqq -2h^2 \sum_{Q_0}\sum w L_h{}^0 w \\
&+ \left(\sum_{R_1} \alpha w^2 - \sum_{R_0} \alpha w^2\right) + h\frac{M}{2}\left(\sum_{R_1} w^2 + \sum_{R_0} w^2\right).
\end{aligned} \tag{4.4}
$$

Now, let $i_0 \leqq i \leqq i_0'$. By summation by parts with respect to j, we get

$$
h^2 \sum_{j=j_0}^{j_0'} (w_t{}^2 + w_{\bar{t}}{}^2) = -2h^2 \sum_{j=j_0}^{j_0'} w w_{t\bar{t}} + (w_{j_1}^2 + w_{j_1'}^2 - w_{j_0}^2 - w_{j_0'}^2)\ . \tag{4.5}
$$

Using the identity

$$hw_{t\bar{t}} \equiv w_t - w_{\bar{t}} = (w_t + w_{\bar{t}}) - 2w_t ,$$

we deduce, for $i_0 \leqq i \leqq i_0'$

$$h^2 \sum_{j=j_0}^{j_0'} ww_{t\bar{t}} = (w_{j_0'}w_{j_1'} - w_{j_0}w_{j_1}) - 2h \sum_{j=j_0}^{j_0'} ww_{\bar{t}}$$
$$\geqq -\frac{1}{2}(w_{j_0}^2 + w_{j_1}^2 + w_{j_0'}^2 + w_{j_1'}^2) - 2h \sum_{j'=j_0}^{j_0'} ww_{\bar{t}} .$$

Taking this inequality into (4.5), we get

$$h^2 \sum_{j=j_1}^{j_0'} (w_t^2 + w_{\bar{t}}^2) \leqq 4h \sum_{j=j_0}^{j_0'} ww_{\bar{t}} + 2(w_{j_1}^2 + w_{j_1'}^2) .$$

Summing from $i = i_0$ to $i = i_1$, we get

$$h^2 \sum\sum_{Q_0} (w_t + w_{\bar{t}}^2) \leqq 4h \sum\sum_{Q_0} ww_{\bar{t}} + 2 \sum_{S_1} w^2 .$$

Multiplying this inequality by $h/2$ and adding (4.4), we get

$$h^2 \sum\sum_{Q_0} \left[A(w) + \bar{A}(w) + \frac{h}{2}(w_t^2 + w_{\bar{t}}^2) \right]$$
$$\leqq -2h^2 \sum\sum_{Q_0} w(L_h^0 w - w_{\bar{t}}) + \left(\sum_{R_1} \alpha w^2 - \sum_{R_0} \alpha w^2 \right)$$
$$+ h\frac{M}{2}\left(\sum_{R_1} w^2 + \sum_{R_0} w^2 \right) + h \sum_{S_1} w^2 .$$

Hence

$$h^2 \sum\sum_{Q} (A(w) + \bar{A}(w)) < 2h^2 \sum\sum_{Q_0} |w||L_h w| \tag{4.6}$$
$$+ \left(\sum_{R_1} \alpha w^2 - \sum_{R_0} \alpha w^2 \right) + h(M/2 + 1)\left(\sum_{S_1} w^2 + \sum_{S_0} w^2 \right).$$

The next step of the proof is to estimate $h^2 \sum\sum_{Q_0} (w_x^2 + w_{\bar{x}}^2)$ in terms of $h^2 \sum\sum_{Q_0} (A(w) + \bar{A}(w))$. We have

$$h^2 \sum\sum_{Q_0} (A(w) + \bar{A}(w)) \equiv B + C + D + E , \tag{4.7}$$

where

$$B = h^2 \sum\sum_{Q_0} (\alpha_{i+1}w_x^2 + \alpha_{i-1}w_{\bar{x}}^2) ,$$

$$C = h^2 \sum\sum_{Q_0} w(\alpha_x w_x + \alpha_{\bar{x}} w_{\bar{x}}) ,$$

$$D = \frac{h^2}{2} \sum\sum_{Q_0} w((\beta w)_x - \beta w_x + (\beta w)_{\bar{x}} - \beta w_{\bar{x}})$$
$$= \frac{h^2}{2} \sum\sum_{Q_0} w(w_{i+1}\beta_x + w_{i-1}\beta_{\bar{x}}) ,$$

$$E = 2h^2 \sum\sum_{Q_0} \gamma w^2 \geqq 0 .$$

Using (4.2) we deduce

$$B \geqq mh^2 \sum_{Q_0}\sum (w_x^{\,2} + w_{\bar{x}}^{\,2}) ,$$

$$|C| \leqq Mh^2 \sum_{Q_0}\sum (|w||w_x| + |w||w_{\bar{x}}|)$$

$$\leqq \frac{M}{\kappa} h^2 \sum_{Q_0}\sum w^2 + \frac{\kappa M}{2} h^2 \sum_{Q_0}\sum (w_x^{\,2} + w_{\bar{x}}^{\,2})$$

for any positive number κ

$$|D| \leqq Mh^2 \sum_{Q_1}\sum w^2 .$$

Using those estimates we deduce from (4.7)

$$(4.8) \qquad \begin{aligned} &(m - \kappa M/2)h^2 \sum_{Q_0}\sum (w_x^{\,2} + w_{\bar{x}}^{\,2}) \\ &\qquad \leqq h^2 \sum_{Q_0}\sum (A(w) + \bar{A}(w)) + M(1 + 1/\kappa)h^2 \sum_{Q_1}\sum w^2 . \end{aligned}$$

Lemma 3.1 follows directly from (4.6) and (4.8) and the obvious fact that the preceding argument is valid for any k and not only for $k = 1$.

We will need also the following:

LEMMA 4.2. *Let $G'' \subset\subset G'$ be an arbitrary interior subdomain of G'. Suppose that $w(P)$ satisfies for any rectangle $Q_k \subset G'(h)$ an inequality of the form*

$$(4.9) \qquad \begin{aligned} h^2 \sum_{Q_{k-1}}\sum (w_x^{\,2} + w_{\bar{x}}^{\,2}) &\leqq M_0\Big(\sum_{R_k} \phi w^2 - \sum_{R_{k-1}} \phi w^2\Big) \\ &+ M_1 h\Big(\sum_{S_k} w^2 + \sum_{S_{k-1}} w^2\Big) + M_2 h^2 \sum_{Q_k}\sum w^2 + M_3 , \end{aligned}$$

where M_0, M_1, M_2, M_3 are positive constants and where $\phi(P)$ is a positive bounded function defined on $G'(h)$.

Then, we have an estimate of the form

$$(4.10) \qquad h^2 \sum_{G''(h)}\sum w_x^{\,2} \leqq Kh^2 \sum_{G'(h)}\sum w^2 + K' ,$$

where the constants K and K' depend only on the constants M_0, M_1, M_2, M_3, on the bound of the function $\phi(P)$ and on the domains G' and G''.

Proof. The proof of this lemma is essentially contained in Courant, Friedrichs and Lewy [1]. It is based on a double summation of inequality (4.3).

Proof of Theorem 4.1. Now that we have Lemmas 4.1 and 4.2, we are able to prove the theorem (in the case $n = 2$). First, we observe that, in the case $n = 2$, the discrete analogue of Sobolev's imbedding theorem is true if we assume only the boundedness of the sums $h^2 \sum\sum_{G'(h)} v_x^2$, $h^2 \sum\sum_{G'(h)} v_t^2$, and $h^2 \sum\sum_{G'(h)} v_{xt}^2$ (see Courant, Friedrichs and Lewy [1]).

We will study separately each of these sums.

(a) *Boundedness of* $h^2 \sum\sum_{G'(h)} v_x^2$ Since $|v(P; h)| < M$ and $|L_h v(P; h)| = |f(P)| < M$, $\forall P \in G'(h)$, it follows from Lemma 4.1 that the function $w = v$ satisfies an inequality of the form (4.9). Applying Lemma 4.2, we deduce that the sums $h^2 \sum\sum_{G''(h)} v_x^2$ are uniformly bounded, where G'', just as G', is an arbitrary interior subdomain of G.

(b) *Boundedness of* $h^2 \sum\sum_{G'(h)} v_{\bar{t}}^2$. Let $w = v_x$. We deduce from (4.1)

$$L_h w = f_x - \alpha_x w_x - \beta_x((w + w_{i+1})/2) + \gamma_x v_{i+1}\,.$$

Therefore

$$h^2 \sum_{G'}\sum_{(h)} |w||L_h w| < Mh^2 \sum_{G'}\sum_{(h)} [|w| + |w||w_x| + \tfrac{1}{2}|w|^2 + \tfrac{1}{2}|w||w_{i+1}| + M]\,.$$

Applying the inequalities

$$\begin{aligned} |w| &\leqq \tfrac{1}{2}(1 + w^2)\,, \\ |w||w_x| &\leqq w^2/2\kappa + \kappa w_x^2/2\,, \\ |w||w_{i+1}| &\leqq w^2/2 + w_{i+1}^2/2\,, \end{aligned}$$

and using the previous result on the boundedness of $h^2 \sum\sum_{G'(h)} w^2$, we deduce

$$h^2 \sum_{G'}\sum_{(h)} |w||L_h w| < \mathfrak{M}(\kappa) + \kappa(M^2/2)h^2 \sum_{G'}\sum_{(h)} |w_x|^2$$

where $\mathfrak{M}(\kappa)$ is some positive constant depending on M and κ. Choose κ such that $m - \kappa(M/2 + M^2) > 0$ and set this estimate into (4.3). We get an inequality of the form (4.9) and therefore we can apply Lemma 4.2 which shows that the sums $h^2 \sum\sum_{G'(h)} v_{xx}^2$ or $h^2 \sum\sum_{G'(h)} v_{x\bar{x}}^2$ are uniformly bounded for any $G' \subset\subset G$. But, (4.1) yields

$$|v_{\bar{t}}| < M|v_{x\bar{x}}| + M(|v_x| + |v_{\bar{x}}|)/2 + M|v| + M\,.$$

Therefore, the boundedness of the sums $h^2 \sum\sum_{G'(h)} v_x^2$ and $h^2 \sum\sum_{G'(h)} v_{x\bar{x}}^2$ implies the boundedness of the sums $h^2 \sum\sum_{G'(h)} v_{\bar{t}}^2$ (or $h^2 \sum\sum_{G'(h)} v_t^2$).

(c) *Boundedness of* $h^2 \sum\sum_{G'(h)} v_{xt}$. Let $w = v_t$. We deduce from (4.1)

$$\begin{aligned} L_h w &= f_t - \alpha_t v_{x\bar{x}} - \beta_t(v_x + v_{\bar{x}})/2 - \gamma_t v \\ &= f_t - (\alpha_t/\alpha)(f - \beta(v_x + v_{\bar{x}})/2 + \gamma v + v_{\bar{t}}) - \beta_t(v_x + v_{\bar{x}})/2 - \gamma_t v\,. \end{aligned}$$

Hence

$$|L_h w| < M(1 + M/m)(1 + |v| + (|v_x| + |v_{\bar{x}}|)/2) + (M/m)|v_{\bar{t}}|\,.$$

Taking this inequality into (4.3) and applying the previous results on the boundedness of the sums $h^2 \sum\sum v_x^2$ and $h^2 \sum\sum v_t^2$ we deduce, as before, an inequality of the form (4.9) which by application of Lemma 4.2 ends the proof of (c) and of the interior continuity of the family $\mathfrak{F}$. The interior equicontinuity of the families $\mathfrak{F}^{(p)}$ is proved in the same way, after differencing the finite-difference equation (4.1) p times.

COROLLARY 4.1. *Same hypotheses as in Theorem* 4.1. *Then, any sequence* $\{v(P; h_n); h_n \to 0\} \subset \mathfrak{F}$ *admits a subsequence which converges uniformly in* G' *to a solution of the differential equation* (1.6).

Proof. For h small enough, G' is covered by cubic cells of the mesh; by linear interpolation in those cells, we can extend the mesh-functions into continuous functions defined on $\overline{G'}$. Thus, an equicontinuous family of mesh-functions is extended into an equicontinuous family of functions defined on all of $\overline{G'}$. The theorem follows by application of Ascoli's theorem to the families $\mathfrak{F}$, $\mathfrak{F}^{(1)}$ and $\mathfrak{F}^{(2)}$ and because of our consistency assumption (2.18).

Remark. We have assumed that the coefficients of the differential equation are in $C^\infty(G)$. Indeed we do not need so much smoothness. The degree of smoothness which is needed in Corollary 4.1 depends on n. In particular, in the case $n = 2$, our proof of Theorem 4.1 shows that the family $\mathfrak{F}$ is equicontinuous in any interior subdomain of G if we assume only Lipschitz-continuity of the coefficients and of the function $f(P)$ in any interior subdomain of G; for the equicontinuity of the families $\mathfrak{F}^{(1)}$ and $\mathfrak{F}^{(2)}$ we need the assumption that the coefficients and the function $f(P)$ admit Lipschitz-continuous derivatives of order 2 in any interior subdomain of G; if we do not assume so much, then we can only prove that the limit function is a weak solution of the differential equation (1.6); to prove this we use the fact that the operator L_h is a weakly consistent (see [6]) approximation to the operator L in any interior subdomain of G. Very general results concerning weak solutions of coercive parabolic problems and their numerical computation can be found in Raviart [10].

V. Existence of Discrete Barriers. Let L and L_h be the operators defined by (2.13) and (2.14). Throughout this section we consider a point $Q = (x_0, y_0, t_0)$ on Γ_1 and we study various types of *local* conditions on G and on L_h which guarantee the existence of a strong discrete barrier at Q. We assume that there exists a neighborhood N of Q such that $G(h) \cap N \subset G_0(h)$ for h small enough.*

(1) *First sufficient condition.* Assume that the coefficients of the operator L are uniformly continuous and that L_h is a uniformly consistent approximation to L in a neighborhood of Q.**

Assume $a(Q) > 0$*** and that there exists a nondegenerate sphere through Q whose intersection with $\overline{G}$ is the single point Q and whose center is not in the plane $x = x_0$. Then, there exists a strong discrete barrier at Q.

Proof. Let us take the origin at the center of the sphere and let

$$s = x^2 + y^2 + t^2\,, \qquad s_0 = s(Q) = x_0^2 + y_0^2 + t_0^2\,.$$

Let k and p be positive constants and $B(P; Q) = k(s^{-p} - s_0^{-p})$. This function obviously satisfies condition (2.8a, b, c). Moreover, we have

$$\begin{aligned} LB(P;Q) &= 2kps^{-p-2}[2(p+1)(ax^2 + a'y^2) - s(a + bx + a' + b'y - dt)] \\ &\quad - cB(P;Q)\,. \end{aligned}$$

In a certain neighborhood N of Q we have $a(P) > \frac{1}{2}a(Q) > 0$, $x^2 > \frac{1}{2}x_0^2$, and therefore

$$LB(P;Q) > 2kps^{-p-2}[\tfrac{1}{2}(p+1)a(Q)x_0^2 - s(a + bx + a' + b'y - dt)]\,.$$

It follows that $LB(P;Q)$ can be made arbitrarily large in N provided we choose k and p big enough. In particular we can choose k and p such that

$$\begin{aligned} L_hB(P;Q) - E(P) &= L_hB(P;Q) - \gamma(P;h) \\ &= LB(P;Q) - c(P) + O(1) > 1 \text{ in } N\,, \qquad \text{for } h \text{ small enough.} \end{aligned}$$

* For instance, this condition holds if $N \cap \overline{\Gamma}_2 = \varnothing$ or if we choose $G(h) = G_0(h)$.

** Observe that this condition is satisfied for the 2 operators L_h corresponding to formulas (2.15) and (2.16).

*** The values of the coefficients at Q are defined by continuity.

Thus, $B(P; Q)$ is a strong discrete barrier at Q.

Remark. Of course, we get a similar condition by permutation of x and y.

(2) *Second sufficient condition.* Assume that the coefficients of the operator L are uniformly continuous and that L_h is a uniformly consistent approximation to L in a neighborhood N of Q. Assume $d(Q) > 0$ and that there exists a nondegenerate sphere through Q with radius $R > (a(Q) + a'(Q))/d(Q)$, whose intersection with $\bar{G} \cap N$ is the single point Q and whose center lies on the half-line $x = x_0$, $y = y_0$, $t < t_0$. Then, there exists a strong discrete barrier at Q.

Proof. Let $B(P; Q)$ be defined as before. Then

$$LB(P;Q) > 2kps^{-p-1}[dt - (a + bx + a' + b'y)] .$$

Since the square bracket tends to $Rd(Q) - a(Q) - a'(Q) > 0$ as $P \to Q$, we see that $LB(P; Q)$ can be made arbitrarily large in a neighborhood of Q provided we choose k and p large enough. It follows as before that $B(P; Q)$ is a strong discrete barrier at Q.

Remark. The condition on the radius R of the sphere is perhaps unnecessary; however, it is related to the results of Kohn and Nirenberg [7] who emphasized the influence of the radius of curvature at a "characteristic" point of the boundary, on the smoothness of the solution.

(3) *Third sufficient condition.* Assume that there exists a neighborhood N of Q such that $G \cap N$ lies in the half-space $t > t_0$. Assume that the coefficients of the operator L are bounded, except $d(P)$ which may be unbounded, $d(P) > k(t - t_0)^\sigma$, $\sigma < 1$, $k > 0$. Let L_h be the operator corresponding to formulas (2.15) or to formulas (2.16). Then, there exists a strong discrete barrier at Q.

Proof. Let us take the origin at Q.

Case 1. Suppose $0 < \sigma < 1$. Let $B(P; Q) = -x^2 - y^2 - Kt^{1-\sigma}$, $K > 0$. Then

$$L_hB(P;Q) = -2(\alpha + \beta x + \alpha' + \beta' y) + Kd\frac{t^{1-\sigma} - (t-h)^{1-\sigma}}{h} - \gamma B(P;Q) .$$

But

$$Kd\frac{t^{1-\sigma} - (t-h)^{1-\sigma}}{h} > Kd(1-\sigma)t^{-\sigma} > Kk(1-\sigma) .$$

It follows that condition (2.8d) is satisfied if we choose K large enough. Then, $B(P; Q)$ is a strong discrete barrier at Q.

Case 2. Suppose $\sigma < 0$. Let $B(P; Q) = -x^2 - y^2 - Kt$, $K > 0$. Then, for K large enough, $B(P; Q)$ is a strong discrete barrier at Q (straightforward).

(4) *Fourth sufficient condition.* Suppose that there exists a neighborhood N of Q such that $G \cap N$ is a cylinder parallel to the t-axis. Let us write $L = L_0 - d(\partial/\partial t)$; L_0 is an elliptic space-operator whose coefficients may depend on t. Define L_h^0 in the same way as L^0, i.e., $L_hv \equiv L_h^0 v - \delta v_{\bar{t}}$.

Suppose that there exists a function $B_0(P; Q)$ which does not depend on t and which is a strong discrete barrier for the family of space-operators L_h^0 for any t such that $|t - t_0| < \tau$, where $\tau > 0$ is a constant (independent of h). Suppose $\delta(P)$ is bounded. Then, there exists a strong discrete barrier at Q for the family of operators L_h.

Proof. $B_0(P; Q)$ satisfies the conditions

$$B_0(P;Q) = B_0(x,y;Q) \in C(\overline{G \cap N}) ,$$
$$B_0(x_0,y_0;Q) = 0 ,$$
$$B_0(x,y;Q) < 0 , \qquad \forall (x,y) \neq (x_0,y_0) ,$$
$$L_h^0 B_0(P;Q) - E(P) \geqq 1 , \qquad \forall P \in N(h) \text{ and } \forall h \text{ small enough} .$$

Let $B(P;Q) = KB_0(P;Q) - (t - t_0)^2$, $K > 1$.

This function satisfies conditions (2.8); therefore it is a strong discrete barrier at Q for the family of operators L_h.

(5) *Applications.* By means of the fourth sufficient condition, all the results of [5] and [6] for elliptic operators L^0 are directly extended to the corresponding parabolic operators $L = L^0 - d(\partial/\partial t)$.

Example 1. Let $\psi(x)$ be a convex function defined for all real x and such that $|\psi(x_1) - \psi(x_2)|/|x_1 - x_2| < M$ for all x_1 and $x_2 \neq x_1$, where M is a positive constant. Let $\mathcal{C}$ be the curve $Y \equiv y - \psi(x) = 0$ in the plane $t = 0$. Let G_0 be a bounded simply-connected plane domain whose boundary consists of a portion of $\mathcal{C}$ and of a smooth curve which lies entirely in the region $Y > 0$. Let $G = G_0 \times (0, T)$ be a cylinder and $G_\epsilon = G \cap \{P = (x, y, t);\ Y > \epsilon\}$. Let $\Gamma_2 = \{P = (x, y, T) \in \partial G\}$ and $\Gamma_1 = \partial G - \Gamma_2$. Let

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + b\frac{\partial}{\partial x} + b'\frac{\partial}{\partial y} - \frac{\partial}{\partial t}, \tag{5.1}$$

where

$$\begin{aligned} &b(P), \quad b'(P) \in C(\overline{G}_\epsilon) \cap C^\infty(G) , \qquad \forall\, \epsilon > 0 ,\\ &[b^2(P) + b'^2(P)]^{1/2} < k/Y + K , \qquad \forall P \in G ,\\ &0 < k < \min\{1, 2/M\} , \qquad K > 0 . \end{aligned} \tag{5.2}$$

Let L_h be the operator defined by formulas (2.14) and (2.15). Conditions (5.2) imply that this operator is of positive type. Let $v(P; h)$ be the solution of (2.9).

THEOREM 5.1. *Under the above hypotheses, problem* (1.7) *has a unique solution* $u(P)$ *and* $v(P; h)$ *converges uniformly to* $u(P)$ *in* G *as* $h \to 0$.

Proof. Let $Q = (x_0, y_0, t_0) \in \partial G$ be such that $(x_0, y_0) \in \mathcal{C}$ and let $B_0(P; Q) = -(x - x_0)^2 - Y^{1-k'}$, where $k < k' < 1$. The function $B_0(P; Q)$ has the properties required for the application of our fourth sufficient condition (see [5, p. 121]). The existence of a discrete barrier at the other points of Γ_1 follows from our first and second sufficient conditions. The existence of a function $\phi(P)$ satisfying condition (i) of Theorem 2.1 follows from our second sufficient condition in Section III. Unicity follows from the maximum principle for parabolic operators (see Lemma 6.1). So, we can apply Theorem 2.1.

Particular cases. $\mathcal{C}$ is the x-axis and

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\sigma}{y}\frac{\partial}{\partial y} - \frac{\partial}{\partial t}, \qquad |\sigma| < 1 , \tag{5.3}$$

or

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\sigma}{y}\frac{\partial}{\partial x} - \frac{\partial}{\partial t}, \qquad |\sigma| < 1 . \tag{5.4}$$

COROLLARY 5.1. *Let G_0 be a "regular"* convex plane domain in the plane $t = 0$ and let $G = G_0 \times (0, T)$. Let $\Gamma_2 = \{P = (x, y, T) \in \partial G\}$ and $\Gamma_1 = \partial G - \Gamma_2$. Let L be the operator* (5.1) *where*

$$(5.5)\qquad \begin{aligned} & b(P), \quad b'(P) \in C^\infty(G)\,, \\ & [b^2(P) + b'^2(P)]^{1/2} < k/d(P, \partial G) + K\,, \qquad \forall P \in G\,, \\ & 0 < k < 1/\sqrt{2}\,, \qquad K > 0\,. \end{aligned}$$

Let L_h be the operator defined by (2.14) *and* (2.15) *and let $v(P; h)$ be the solution of* (2.9). *Then, problem* (1.7) *has a unique solution $u(P)$ and $v(P; h)$ converges uniformly to $u(P)$ in G as $h \to 0$.*

Proof. Same as for Theorem 5.1.

Example 2. Let G be the same domain as in the particular cases above. Let

$$(5.6)\qquad L = y\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\sigma}{y}\frac{\partial}{\partial y} - \frac{\partial}{\partial t}\,, \qquad |\sigma| < 1\,,$$

or

$$(5.7)\qquad L = \frac{1}{y}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\sigma}{y}\frac{\partial}{\partial y} - \frac{\partial}{\partial t}\,, \qquad |\sigma| < 1\,.$$

And let L_h be either of the operators defined by formulas (2.14) and (2.15) or by formulas (2.14) and (2.16). Then, the conclusion of Theorem 5.1 holds.

This is a direct consequence of our fourth sufficient condition and of the results of [6, Theorems 4.1–4.4].

Remark. The preceding conditions for the existence of discrete barriers are only examples; we can imagine many other conditions; it seems impossible to gather all these conditions in a unique general condition.

VI. Unicity. Again G is a domain in R^3 and L is the operator (2.13). We denote by Γ' the set of all points $Q = (x_0, y_0, t_0) \in \partial G$ which admit a neighborhood N such that $\partial G \cap N$ lies in the plane $t = t_0$ and $G \cap N$ lies in the half-space $t < t_0$; Γ' is called the set of "final" points of G.

LEMMA 6.1. *Suppose $\Gamma_2 \subset \Gamma'$. Then problem* (1.7) *has at most one solution.*

Proof. By the maximum principle.

We deduce at once the following

COROLLARY 6.1. *A necessary condition for the existence of a solution of problem* (1.7) *for arbitrary $g(P) \in C(\overline{G})$ is $\Gamma' \subset \Gamma_2$.*

From now on we will assume $\Gamma' \subset \Gamma_2$ and we define $\Gamma'' = \Gamma_2 - \Gamma'$. The following lemma is a generalization of an idea which has been used by S. V. Parter [9, §4] for generalized axially symmetric potentials.

LEMMA 6.2. *Suppose Γ'' is closed and suppose that there exists a neighborhood N of Γ'' and a function $U(P)$ such that*

$$(6.1)\qquad \begin{aligned} & U \in C^2(G_0) \cap C(\overline{G}_0 - \Gamma'') \qquad \text{where } G_0 = G \cap N\,, \\ & LU(P) \leqq 0\,, \qquad P \in G_0\,, \\ & U(P) \to +\infty \text{ as } P \to Q\,, \qquad \forall Q \in \Gamma''\,, \qquad P \in \overline{G}_0 - \Gamma''\,. \end{aligned}$$

* By "regular" we mean that in the neighborhood of any point $Q_0 \in \partial G_0$, ∂G_0 admits a representation of the form $y = \phi(x)$ or of the form $x = \psi(y)$ where ϕ and ψ are convex functions.

Then, problem (1.7) has at most one solution.

Proof. We can, of course, suppose $U(P) > 0$ since it is always possible to make it so by addition of a sufficiently large positive constant. Let $z(P)$ be a solution of the homogeneous problem associated to (1.7), i.e.

$$\begin{aligned} &Lz(P) = 0\,, \qquad P \in G\,, \\ &z(P) = 0\,, \qquad P \in \Gamma_1\,, \\ &z \in C^2(G) \cap C(G \cup \Gamma_1) \cap B(G)\,. \end{aligned} \tag{6.2}$$

Let $C_0 = \operatorname{Sup}_{P\in G-G_0} z(P)$ and suppose $C_0 > 0$. Let ∂G_0 be the boundary of G_0. It follows from the maximum principle that there exists $P_0 \in \partial G_0 \cap G$ such that $z(P_0) = C_0$. Let $C_1 = \operatorname{Sup}_{P\in G_0} z(P)$. Let N' be an arbitrary neighborhood of Γ'' such that $\overline{N}' \subset N$ and let $C_2 = \operatorname{Sup}_{P\in G_0-N'} U(P)$.

Let α be a positive number and $U_\alpha = C_0 + \alpha U$. For any $\alpha > 0$, there exists a neighborhood of Γ'', $N'' \subset N'$ such that $U_\alpha(P) > C_1$ in $N'' \cap G$. Let $G_1 = G_0 - N''$ and let ∂G_1 be the boundary of G_1. It follows from the definitions of C_0 and C_1 that $z(P) \leqq U_\alpha(P)$ on $\partial G_1 - \Gamma'$. Therefore, by the maximum principle $z(P) \leqq U_\alpha(P)$ in G_1. In particular, by definition of C_2 $z(P) \leqq U_\alpha(P) \leqq C_0 + \alpha C_2$ in $G_0 - N'$.

Since α is arbitrary, we deduce $z(P) \leqq C_0$ in $G_0 - N'$ and since N' is arbitrary, $z(P) \leqq C_0$ in G_0. Hence, by definition of C_0 $z(P) \leqq C_0$ in G.

But at $P_0 \in G$, we have $z(P_0) = C_0$. Therefore, by the maximum principle $z(P) \equiv C_0 > 0$ in G. This is a contradiction of (6.2) since $z(P) = 0$ on Γ_1. Therefore, we must have $C_0 \leqq 0$, which implies $z(P) \leqq 0$ in G, since N can be arbitrarily small. We deduce the reverse inequality in the same way and finally $z(P) \equiv 0$, which ends the proof of the lemma.

THEOREM 6.1. *Let $G_1, G_2, \cdots, G_r, \cdots, G_n$ be a finite partition of G into subdomains of the form $G_r = G \cap I_r$ where I_r is a slab $t_r < t < t_{r+1}$. Let Γ_r'' be the closure of $\Gamma'' \cap I_r$ and suppose that for each r there exists a neighborhood N_r of Γ_r'' and a function $U_r(P)$ such that*

$$\begin{aligned} &U_r \in C^2(G_r^0) \cap C(\overline{G}_r^0 - \Gamma_r'') \qquad \text{where } G_r^0 = G_r \cap N_r\,, \\ &LU_r(P) \leqq 0\,, \qquad P \in G_r^0\,, \\ &U_r(P) \to +\infty \text{ as } P \to Q\,, \qquad \forall Q \in \Gamma_r''\,, \qquad P \in \overline{G}_r^0 - \Gamma_r''\,. \end{aligned} \tag{6.3}$$

Then, problem (1.7) has at most one solution.

Proof. Apply Lemmas 6.1 and 6.2.

Now, we give an example of application of Theorem 6.1.

THEOREM 6.2. *Suppose G lies in the half-space $x > 0$ and let L be the operator* (2.13). *Let I be a slab $t_1 < t < t_2$ and assume that there exists a constant $K \geqq 0$ such that $b(P)/a(P) > 1/x - K$ for all $P = (x, y, t) \in G \cap I$, x small enough.*

Let $\Gamma'' = \partial G \cap \bar{I} \cap \{P = (0, y, t)\}$. Then, problem (1.7) has at most one solution.

Proof. Let $U(P) = -Kx - \operatorname{Log} x$. We have

$$\begin{aligned} LU(P) &= a(P)/x^2 - b(P)(K + 1/x) \\ &< a(P)[1/x^2 - (1/x - K)(K + 1/x)] \\ &= -K^2 a(P) \leqq 0 \qquad \text{in } G \cap I \text{ and for } x \text{ small enough}\,. \end{aligned}$$

Therefore, the assumptions of Theorem 6.1 are satisfied.

Examples.

$$Lu \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\sigma(t)}{x}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}$$

or

$$Lu \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + [x^{-\sigma(t)}]\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}$$

with $\sigma(t) \geqq 1$ for $t_1 \leqq t \leqq t_2$.

The following theorem is closely related to Theorem 6.1. It can be proved in the same way.

THEOREM 6.3. *Suppose G lies in the half-space $t > 0$ and let Γ_0 be the portion of the boundary ∂G which lies in the plane $t = 0$. Let L be the operator* (2.13) *and let $\Gamma'' = \Gamma_0$. Assume that there exists a neighborhood N of Γ_0 and a function $U(P)$ such that*

$$U \in C^2(G^0) \cap C(\overline{G^0} - \Gamma_0)\,, \qquad \text{where } G^0 = G \cap N\,,$$
$$LU(P) \leqq 0\,, \qquad P \in G^0\,,$$
$$U(P) \to +\infty \text{ as } P \to Q\,, \qquad \forall Q \in \Gamma_0\,, \qquad P \in \overline{G^0} - \Gamma_0\,.$$

Then, problem (1.7) *has at most one solution.*

Example. See Section 7, Example 2.

TABLE I

$N = 1/h$	$v_1(P, h)$	$v_2(P, h)$	$v_3(P, h)$	$v_4(P, h)$
4	0.3245	0.3302	0.3231	0.3270
8	260	305	275	279
16	274	307	295	288
32	285	309	304	294
64	292	310	308	299
128	297	310	309	302
256	300	310	310	304
512	303	311	310	306
$u(P)$	0.3311			

VII. Numerical Experiments.

(1) *First example.* First, we study the example given in the introduction. G is the triangle $0 < x < t < 2$ and

$$\Gamma_1 = \{P = (x, t); 0 \leqq t = x \leqq 2\} \cup \{P = (0, t); 0 \leqq t < 1\}\,.$$

We consider the problem

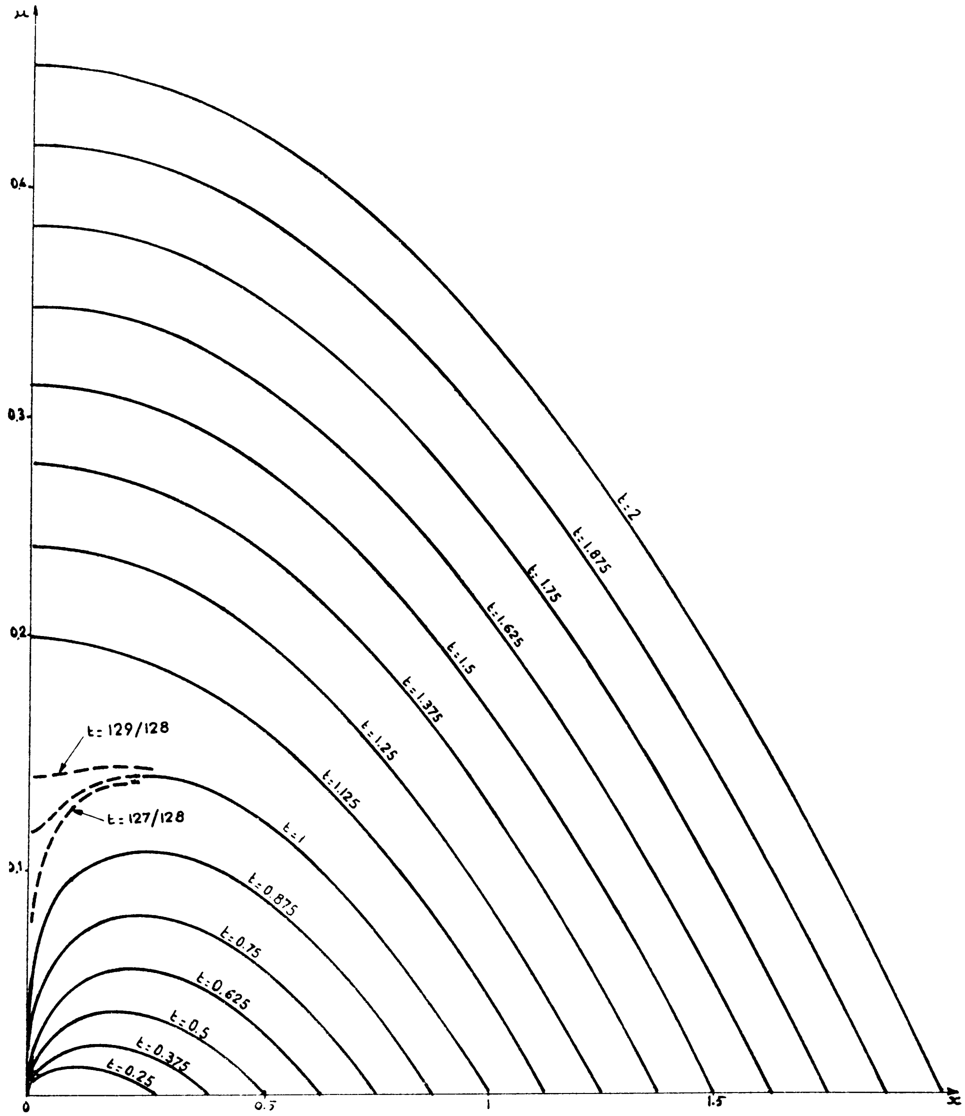
$$Lu \equiv \frac{\partial^2 u}{\partial x^2} + \frac{t}{x}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = -1 \quad \text{in } G\,,$$
$$u = 0 \quad \text{on } \Gamma_1\,, \tag{7.1}$$
$$u \in C^2(G) \cap C(G \cup \Gamma_1) \cap B(G)\,.$$

The uniqueness of the solution follows from Theorem 6.2. To compute this solution (and prove its existence) we will consider four different schemes. Let $h = 1/N$, N integer; we define

$$\begin{aligned}
R(h) &= \{P = (ih, jh);\, i, j \text{ integers}\}\,, \\
\overline{G}(h) &= \overline{G} \cap R(h)\,, \\
\Gamma_1(h) &= \Gamma_1 \cap R(h)\,, \\
\Gamma_2(h) &= \Gamma'' \cap R(h) \qquad \text{where } \Gamma'' = \{P = (0, t);\, 1 \leqq t \leqq 2\}\,, \\
G_0(h) &= \{P = (x, t);\, 0 < x < t \leqq 2\} \cap R(h)\,.
\end{aligned}$$

Thus, $\overline{G}(h) = G_0(h) \cup \Gamma_1(h) \cup \Gamma_2(h)$.

TABLE II



At each point $P \in G_0(h)$ we define

$$L_h v(P) \equiv v_{x\bar{x}} + (t/x)(v_x + v_{\bar{x}})/2 - v_{\bar{t}} .$$

Scheme 1. We take $G(h) = G_0(h)$, $\partial G(h) = \Gamma_1(h) \cup \Gamma_2(h)$, $g(P) = 0$ on $\Gamma_2(h)$. Then, the discrete analogue of problem (7.1) is

$$\begin{aligned} L_h v(P) &= -1 , \quad P \in G_0(h) , \\ v(P) &= 0 , \quad P \in \Gamma_1(h) \cup \Gamma_2(h) . \end{aligned} \tag{7.2}$$

Scheme 2. We take $G(h) = G_0(h) \cup \Gamma_2(h)$, $\partial G(h) = \Gamma_1(h)$, $L_h v(P) \equiv v_x$ if $P \in \Gamma_2(h)$, $f(P) = 0$ on $\Gamma_2(h)$. Then, the discrete analogue of problem (7.1) is

$$\begin{aligned} L_h v(P) &= -1 , \qquad P \in G_0(h) , \\ v(P) &= 0 , \qquad P \in \Gamma_1(h) , \\ v_x(P) &= 0 , \qquad P \in \Gamma_2(h) . \end{aligned} \tag{7.3}$$

Scheme 3. Same as Scheme 2, except that we take $f(P) = 1$ on $\Gamma_2(h)$. Then, we have

$$\begin{aligned} L_h v(P) &= -1 , \qquad P \in G_0(h) , \\ v(P) &= 0 , \qquad P \in \Gamma_1(h) , \\ v_x(P) &= 1 , \qquad P \in \Gamma_2(h) . \end{aligned} \tag{7.4}$$

Scheme 4. We take $G(h) = G_0(h) \cup \Gamma_2(h)$, $\partial G(h) = \Gamma_1(h)$, $L_h v(P) = v_{1j} - 2v_{0j}$ if $P = (0, jh) \in \Gamma_2(h)$, $f(P) = 0$ on $\Gamma_2(h)$. Then, we have

$$\begin{aligned} L_h v(P) &= -1 , \qquad P \in G_0(h) , \\ v(P) &= 0 , \qquad P \in \Gamma_1(h) , \\ v_{1j} - 2v_{0j} &= 0 , \qquad P = (0, jh) \in \Gamma_2(h) . \end{aligned} \tag{7.5}$$

The four schemes are of positive type. The function $\phi(P) = x - t - 1$ satisfies condition (i) of Theorem 2.1; the existence of discrete barriers at the points of Γ_1 follows from the first and fourth conditions of Section V. Therefore, we can apply Theorem 2.1. The four schemes (7.2), (7.3), (7.4) and (7.5) converge to the unique solution of problem (7.1), uniformly in any $G_\epsilon = G - \{P = (x, t); t > 1 - \epsilon, 0 < x < \epsilon\}$.*

Table I shows the convergence at the point $P(1, 2)$ of the functions $v_s(P; h)$, $s = 1, 2, 3, 4$, corresponding to each of the foregoing schemes.**

It appears that Scheme 2 is the best; this is related to the observed fact that the solution $u(P)$ satisfies $\partial u/\partial x = 0$ on Γ''. A closer examination of the results shows that the convergence of this scheme is uniform in G except for a neighborhood of the point $(1, 1)$; of course, we cannot expect better than that since $u(P)$ is not continuous at this point.

* A direct application of Theorem 2.1 requires that we exclude also a neighborhood of the line $t = 2$. But, of course, we can extend the domain G for $t > 2$ in such a way that the operator remains of positive type and the "final" points of G (on the line segment $t = 2$, $0 < x < 2$) become interior points. Applying Theorem 2.1 to this extended domain, we deduce that the convergence in the domain G is uniform up to $t = 2$.

** The author is indebted to Mrs. F. Glain for the numerical computations.

Scheme 3 is not as good as Scheme 2; but the convergence is again uniform in G except for a neighborhood of the point (1, 1), despite the fact that we try to impose a wrong condition on the derivative $\partial u/\partial x$ on Γ''.

Schemes 1 and 4 converge also, but the convergence is not uniform in the neighborhood of Γ''; in Scheme 1 we try to impose wrong values to the function u on Γ''; in Scheme 4 we use a meaningless condition.*

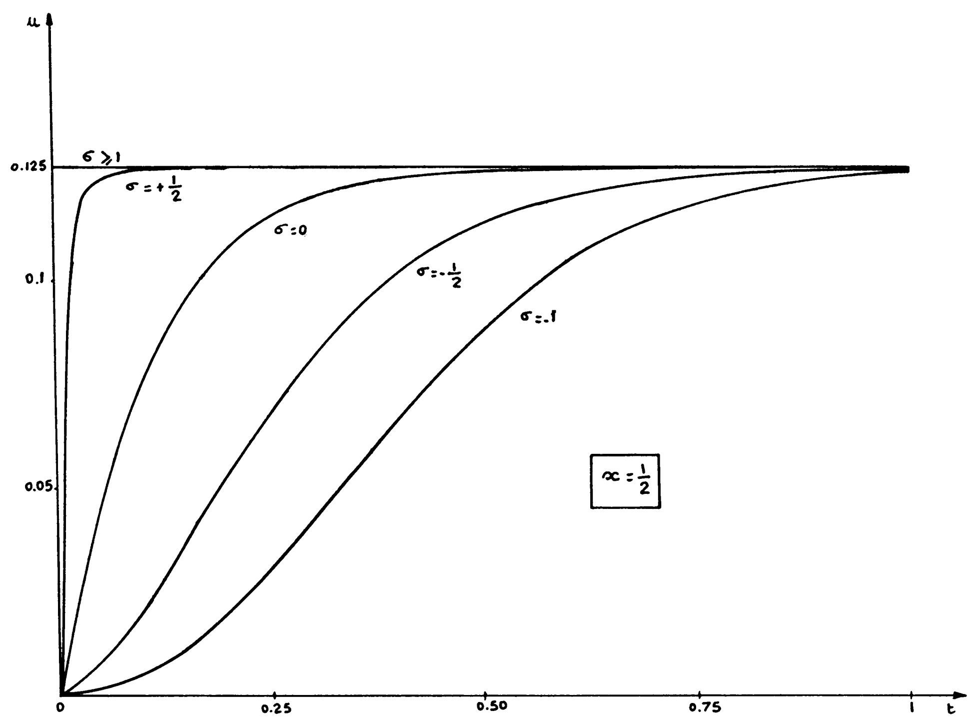
Table II represents the solution $u(P)$. The values of $u(P)$ are not known accurately near the point (1, 1) where this function is discontinuous

(2) *Second example.* Let G be the rectangle: $0 < x < 1$, $0 < t < T$, where T is some positive number, and let Γ_1 be the three sides of the rectangle: $x = 0$, $x = 1$ and $t = 0$. Let σ be a real number. We consider the problem

$$
(7.6) \qquad
\begin{aligned}
&Lu \equiv \frac{\partial^2 u}{\partial x^2} - t^\sigma \frac{\partial u}{\partial t} = -1 \quad \text{in } G\,,\\
&u = 0 \text{ on } \Gamma_1\,,\\
&u \in C^2(G) \cap C(G \cup \Gamma_1) \cap B(G)\,.
\end{aligned}
$$

We define $R(h)$ as usual, $\Gamma_1(h) = \Gamma_1 \cap R(h)$, $G(h) = (\bar{G} - \Gamma_1) \cap R(h)$ and $L_h v(P) = v_{x\bar{x}} - t^\sigma v_{\bar{t}}$.

TABLE III



* Other types of finite-difference schemes with "wrong" boundary conditions have been studied by S. V. Parter [9′] and by H.-O. Kreiss and E. Lundqvist [8′].

The discrete analogue of problem (7.6) is

$$L_h v(P) = -1, \quad P \in G(h),$$
$$(7.7) \qquad v(P) = 0, \quad P \in \Gamma_1(h).$$

This scheme is of positive type; the function $\phi(P) = x^2$ satisfies condition (i) of Theorem 2.1; if $\sigma < 1$, we can apply our third sufficient condition for the existence of barriers (Section 5); if $\sigma \geqq 1$, we can apply Theorem 6.3 with $U(P) = -x^2 - \operatorname{Log} t$. It follows that

if $\sigma < 1$, problem (7.6) has a unique solution;

if $\sigma \geqq 1$, problem (7.6) has no solution: a solution of the differential equation is uniquely determined by the boundary-values at $x = 0$ and $x = 1$; *we can impose no initial condition.*

Table III represents the solution as a function of t for $x = \frac{1}{2}$ and for $\sigma = \frac{1}{2}, 0, -\frac{1}{2}, -1$. When $t \to \infty$, $u(x, t) \to \frac{1}{2}x(1 - x)$.

For $\sigma \geqq 1$, the solution determined by the boundary values alone is: $u(x, t) \equiv \frac{1}{2}x(1 - x)$.

TABLE IV

$N = 1/h$	$V(P, h)$
16	0.12181582
32	0.12489043
64	0.12499968
128	0.12500000
256	0.12500000

The numerical experiments show that the convergence is of the order of h for $\sigma < 1$. In the case $\sigma \geqq 1$, the convergence is incredibly fast even though we start with wrong initial values; this fact is illustrated by Table IV which gives the values computed at $x = \frac{1}{2}$, $t = 1/16$ in the case $\sigma = 1.5$ for different values of $h = 1/N$.

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