# On a Theorem of Piatetsky-Shapiro and Approximation of Multiple Integrals

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**Abstract.** Let $f$ be a function of $s$ real variables which is of period 1 in each variable, and let the integral $I$ of $f$ over the unit cube in $s$-space be approximated by

$$Q(f) = \frac{1}{N} \sum_{r=1}^{N} f(r\mathbf{x})$$

(where $\mathbf{x} = \mathbf{x}(N)$ is a point in $s$-space). For certain classes of $f$'s, defined by conditions on their Fourier coefficients, it is shown using methods of N. M. Korobov, that $\mathbf{x}$'s can be found for which error bounds of the form $|I(f) - Q(f)| < K(f)N^{-p}$ will be true. However, for the class of all $f$'s with absolutely convergent Fourier series, it is shown that there are no $\mathbf{x}$'s for which a bound of the form $|I(f) - Q(f)| = O(F(N))$ will hold, for any $F(N)$ which approaches zero as $N$ goes to infinity. ■

In his book *Number-Theoretic Methods of Approximate Analysis*, N. M. Korobov quotes the following result of I. I. Piatetsky-Shapiro [1]:

THEOREM. *Let $A_s$ denote the class of all functions of s real variables that have period* 1 *in each variable and have an absolutely convergent Fourier series*:

$$f(\mathbf{x}) = \sum_{m_1, \ldots, m_s = -\infty}^{\infty} c(\mathbf{m}) \exp (\mathbf{x} \cdot \mathbf{m}) \tag{1}$$

(*boldface letters denote s-tuples of real numbers*; $\exp a = e^{2\pi i a}$). *Then for any $f \in A_s$ and any positive integer $N$ there is a $\boldsymbol{\theta}$ such that*

$$\left| \int_0^1 \cdots \int_0^1 f(\mathbf{x}) d\mathbf{x} - \frac{1}{N} \sum_{r=1}^{N} f(r\boldsymbol{\theta}) \right| < C \frac{\log N}{N} \tag{2}$$

*where $C = C(f)$.*

Though Korobov takes up this theorem in connection with methods of approximate evaluation of multiple integrals, the theorem itself does not provide such a method, as $\boldsymbol{\theta}$ depends on $f$. The question then arises whether a $\boldsymbol{\theta}(N)$ exists which will make (2) true for all $f \in A_s$. We answer this in the negative; but we do show that there are $\boldsymbol{\theta}$'s which allow a stronger statement than (2) for some considerable subsets of $A_s$.

We will denote the unit cube in $s$-dimensional Euclidean space by $G_s$.

THEOREM 1. *If $N_1, N_2, \cdots$ is an increasing sequence of positive integers, $\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \cdots$ a sequence of s-tuples of real numbers, and $F(n)$ any positive decreasing function such that $F(n) \to 0$ as $n \to \infty$, then there is an $f \in A_s$ such that*

$$\left| \int_{G_s} f - \frac{1}{N_i} \sum_{r=1}^{N_i} f(r\boldsymbol{\theta}^{(i)}) \right| \Big/ F(N_i) \tag{3}$$

*is unbounded as $i \to \infty$.*

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*Proof.* $A_s$ is a Banach space, if, with the expansion (1), we define

$$\|f\| = \sum_{m_1,\ldots,m_s=-\infty}^{\infty} |c(\mathbf{m})| .$$

Define the linear functional $L_i$, $i = 1, 2, \cdots$, by

$$L_i(f) = \frac{1}{F(N_i)}\left(\int_{G_s} f - \frac{1}{N_i}\sum_{r=1}^{N_i} f(r\theta^{(i)})\right).$$

If the theorem does not hold, then $|L_i(f)| \leqq C(f)$, $i = 1, 2, \cdots$ for every $f \in A_s$, where $C(f)$ is some real number. Then by the Banach-Steinhaus Theorem (see, e.g., [2]) there is a constant $K$ such that

$$|L_i(f)| \leqq K\|f\| \tag{4}$$

for all $i$ and all $f \in A_s$. But if we choose $\mathbf{m}^{(i)}$, for each $i$, in such a manner that $\mathbf{m}^{(i)}\cdot\boldsymbol{\theta}^{(i)}$ is within $1/2N_i^2$ of an integer (and the components of $\mathbf{m}^{(i)}$ are not all zero), and set $f_i(\mathbf{x}) = \exp(\mathbf{x}\cdot\mathbf{m}^{(i)})$, then

$$|L_i(f_i)| = \frac{1}{N_i F(N_i)}\left|\sum_{r=1}^{N_i} \exp\,(r^{(i)}\cdot m^{(1)})\right| \geqq \frac{1}{2F(N_i)} ,$$

contradicting (4).

If

$$D = \sum_{m_1,\ldots,m_s=-\infty}^{\infty} d(\mathbf{m})$$

is a convergent ($s$-tuple) series of positive constants, we shall denote by $A_s(D)$ the subset of $A_s$ consisting of those functions having expansions (1) satisfying

$$|c(\mathbf{m})| \leqq Cd(\mathbf{m}) , \qquad -\infty < m_1, \cdots, m_s < \infty \tag{5}$$

for some number $C$.

THEOREM 2. *If $D$ is any convergent s-tuple series of positive numbers, and $N$ is a prime number, then there are integers $a_1, a_2, \cdots, a_s$ between 0 and $N-1$ such that*

$$\left|\int_{G_s} f - \frac{1}{N}\sum_{r=1}^{N} f\left(r\frac{a_1}{N}, r\frac{a_2}{N}, \cdots, r\frac{a_s}{N}\right)\right| < \frac{K(f)}{N} \tag{6}$$

*for all $f \in A_s(D)$.*

*Proof.* Using the expansion (1), we see that $\int_{G_s} f = c(0, \cdots, 0)$ while

$$\begin{aligned} \frac{1}{N}\sum_{r=1}^{N} f\left(\frac{r}{N}a\right) &= \sum_{m_1,\ldots,m_s=-\infty}^{\infty} c(\mathbf{m})\left(\frac{1}{N}\sum_{r=1}^{N}\exp\left(\frac{r}{N}\mathbf{a}\cdot\mathbf{m}\right)\right) \\ &= c(0, \cdots, 0) + \sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}' c(\mathbf{m})\delta_N(\mathbf{a}\cdot\mathbf{m}) \end{aligned} \tag{7}$$

where $\delta_N(n)$ is 1 if $N$ divides $n$ and is 0 otherwise, and the prime on the sum indicates that the term with $m_1 = m_2 = \cdots = m_s = 0$ is omitted. Therefore

$$\begin{aligned} \left|\int_{G_s} f - \frac{1}{N}\sum_{r=1}^{N} f\left(\frac{r}{N}\mathbf{a}\right)\right| &\leqq \sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}' |c(\mathbf{m})|\delta_N(\mathbf{a}\cdot\mathbf{m}) \\ &\leqq C \sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}' d(\mathbf{m})\delta_N(\mathbf{a}\cdot\mathbf{m}) . \end{aligned} \tag{8}$$

Let us now look at the average, for given $N$ and $\mathbf{m}$, of $\delta_N(\mathbf{a}\cdot\mathbf{m})$ over all $s$-tuples $\mathbf{a}$ of integers from 0 to $N-1$. Choosing a $j$ such that $m_j \neq 0$, we see that for any choice of $a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_s$ there is just one value of $a_j$ making $\delta = 1$—since $N$ is a prime—and $N-1$ values for which $\delta = 0$. Thus for each $\mathbf{m}$,

$$\text{av}\, \delta_N(\mathbf{a}\cdot\mathbf{m}) = 1/N .$$

It follows that

$$\min_{0\leqq a_1,\ldots,a_s\leqq N-1}\left|\int_{G_s} f - \frac{1}{N}\sum_{r=1}^{N} f\left(\frac{r}{N}\mathbf{a}\right)\right| \leqq \text{av}\, C \sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}' d(\mathbf{m})\delta_N(\mathbf{a}\cdot\mathbf{m}) \leqq \frac{C}{N}\sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}' d(\mathbf{m}) ,$$

proving the theorem.

This result has consequences for the numerical integration of functions satisfying certain stronger conditions. If, following Korobov, we set

$$\overline{m} = \max(|m|, 1) , \qquad \|\mathbf{m}\| = \overline{m}_1\cdot\overline{m}_2\cdot \cdots \cdot\overline{m}_s ,$$

and denote by $E_s^\alpha$, for $\alpha > 1$, the set of functions having an expansion (1) that satisfies $|c(\mathbf{m})| \leqq C(f)\|\mathbf{m}\|^{-\alpha}$, we have

COROLLARY 1. *For each prime number $P$ and for any positive number $\epsilon$ there are integers $a_1, a_2, \cdots, a_s$ such that for any $f \in E_s^\alpha$*

$$\left|\int_{G_s} f - \frac{1}{P}\sum_{r=1}^{P} f\left(\frac{r}{P}\mathbf{a}\right)\right| < \frac{K(f)}{P^{\alpha-\epsilon}} . \tag{9}$$

*Proof.* Set $\beta = \max(\alpha - \epsilon, 1)$ and set

$$g(\mathbf{x}) = \sum_{m_1,\ldots,m_s=-\infty}^{\infty} \|\mathbf{m}\|^{-\alpha/\beta} \exp(\mathbf{m}\cdot\mathbf{x}) ;$$

and let $\mathbf{a}$ be the $s$-tuple of Theorem 2. Since $\sum t^\beta \leqq (\sum t)^\beta$ whenever $\beta \geqq 1$ and the $t$'s are nonnegative, the quantity

$$\sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}' |c(\mathbf{m})|\delta_P(\mathbf{a}\cdot\mathbf{m})$$

for $f$ is no greater than $C(f)$ times the $\beta$th power of the same quantity for $g$; and the latter is less than or equal to $K(g)/P$.

Korobov obtains a sharper result than this; where we have $P^{-\alpha+\epsilon}$ in (9) he has $P^{-\alpha}\log^\beta P$ for certain values of $\beta$. If we further restrict the class of functions we obtain a result that does not follow directly from Korobov's theorems:

Let $L_s^\alpha$, for $\alpha > 1$, be the class of all functions having an expansion (1) that satisfies

$$|c(\mathbf{m})| \leqq C(\|\mathbf{m}\| \log^{1+\epsilon} \|\mathbf{m}\|)^{-\alpha}$$

for some $C = C(f)$ and $\epsilon = \epsilon(f) > 0$. Then we have, by a proof similar to that of the above corollary,

COROLLARY 2. *For each prime number $P$ there is a set of integers $a_1, \cdots, a_s$ such that for any $f \in L_s^\alpha$*

$$\left| \int_{G_s} f - \frac{1}{P} \sum_{r=1}^{P} f\left(\frac{r}{P}\mathbf{a}\right) \right| < \frac{K(f)}{P^\alpha}.$$

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1. N. M. KOROBOV, *Number-Theoretic Methods of Approximate Analysis*, Fizmatgiz, Moscow, 1963, p. 85. (Russian) MR **28** #716.
2. G. F. SIMMONS, *Introduction to Topology and Modern Analysis*, McGraw-Hill, New York, 1965, p. 239.