Some Factors of the Numbers $G_n = 6^{2^n} + 1$ and $H_n = 10^{2^n} + 1$

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Abstract. All numbers $G_n = 6^{2^n} + 1$ and $H_n = 10^{2^n} + 1$ are searched for factors of the form $p = u \cdot 2^s + 1 < 3.88 \cdot 10^{11}$ for $s \geqq n + 1$, and odd u. The search limit for u was 60000 for G_n, and 156250 for H_n. A number of factors are found in this range. The numbers G_6 and H_6, lacking small factors, are proved composite by calculating $5^{(G_6-1)/2} \pmod{G_6}$ and $3^{(H_6-1)/2} \pmod{H_6}$, the residues found being different from ± 1. The smallest numbers G_n and H_n with unknown characters are G_{11} and H_{10}. ■

In analogy to the Fermat numbers $F_n = 2^{2^n} + 1$, the numbers $A_n = a^{2^n} + 1$ do not possess any algebraic factors, unless $a = b^k$, $k \neq 2^t$. It might thus happen that a number of this form is a prime. A simple way to investigate the primality of A_n is to search for small factors of A_n, at least if a factor is found. The author has undertaken such a search for the numbers $G_n = 6^{2^n} + 1$ and $H_n = 10^{2^n} + 1$. Because of Legendre's theorem, only primes of the form $p = u \cdot 2^s + 1$, with $s \geqq n + 1$ and u odd, need to be tried as factors of A_n. All $p = u \cdot 2^s + 1 < 3.88 \cdot 10^{11}$, with $u < 60000$ for G_n, and with $u < 156250$ for H_n, were tested as factors in all G_n and all H_n. (Since

TABLE 1. *Factors* $p = u \cdot 2^s + 1$ *of* $G_n = 6^{2^n} + 1$.

n	u	s	p
0	—	—	G_0 is prime
1	—	—	G_1 is prime
2	—	—	G_2 is prime
3	1	4	17
3	6175	4	98801
4	11	5	353
4	53	5	1697
4	4599	10	4709377
5	43	6	2753
5	2275	6	145601
5	155117027389401	7	19854979505843329
6	—	—	G_6 is composite
7	1	8	257
7	2983	8	763649
8	11	9	5633
9	79	10	80897
9	1641	11	3360769
10	45903	13	376037377
15	1	16	65537
19	13	20	13631489
25	37	26	2483027969
25	1137	27	152605556737
27	193	28	51808043009

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$n \leqq s - 1$, and s is bounded, this means, of course, only a finite number of numbers A_n.) The results are given in the following Tables 1 and 2, which also include previously known factors for small values of n.

As a result of Table 1, $G_0 - G_2$ are primes, $G_3 - G_5$ are completely factored, and $G_{11} = 6^{2048} + 1$ is the smallest G_n with unknown character. Any factor of G_{11} must be > 245760000.

TABLE 2. *Factors* $p = u \cdot 2^s + 1$ *of* $H_n = 10^{2^n} + 1$

n	u	s	p
0	—	—	H_0 is prime
1	—	—	H_1 is prime
2	9	3	73
2	17	3	137
3	1	4	17
3	367647	4	5882353
4	11	5	353
4	7	6	449
4	5	7	641
4	11	7	1409
4	2183	5	69857
5	155	7	19841
5	15253	6	976193
5	96679	6	6187457
5	6518964113895	7	834427406578561
6	—	—	H_6 is composite
7	1	8	257
7	15	10	15361
7	1771	8	453377
8	21	9	10753
8	16121	9	8253953
9	1479	10	1514497
12	56021	13	458924033
15	1	16	65537
15	11	19	5767169
16	63	17	8257537
17	335	19	175636481
18	305	21	639631361
19	67	20	70254593
19	101439	21	212733001729
20	5	25	167772161
26	17	27	2281701377
29	49	30	52613349377
29	135	31	289910292481

As a result of Table 2, H_0 and H_1 are primes, $H_2 - H_5$ are completely factored, and $H_{10} = 10^{1024} + 1$ is the smallest H_n with unknown character. Any factor of H_n is > 320000000. Lacking small factors, G_6 and H_6 had to be investigated by other means. We thus calculated

$$5^{(G_6-1)/2} \equiv 450\,3205343452\,5551224422\,3550543120\\ 1154045341\,2420512003\,3225131314 \pmod{G_6},$$

and

$$3^{(H_6-1)/2} \equiv 9006\ 5795547782\ 8715847687\ 7626890521\ 4525000218\ 9858344257\ 6923855471 \pmod{H_6},$$

and thus G_6 and H_6 are composite, since the residues $\not\equiv \pm 1$. The residue for G_6 above is given in the number system with the base $= 6$. The primality of the large factors of G_5 and H_5 was established by trying all factors $64k + 1$ smaller than the square root of these numbers.

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