Rational Chebyshev Approximations for the Error Function*

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Abstract. This note presents nearly-best rational approximations for the functions erf (x) and erfc (x), with maximal relative errors ranging down to between 6×10^{-19} and 3×10^{-20}.

In [1] Hart, et al., present rational approximations for the function

$$\operatorname{erfc}(x) \equiv 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

valid for $0 \leqq x \leqq \alpha$, where $\alpha = 4, 8, 10$, or 20. They carefully point out [1, p. 138] that these approximations are not useful for computing the error function

$$\operatorname{erf}(x) \equiv 1 - \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

for small x because of subtraction error, but they do not provide any alternative. Hastings' [2] approximations for erf (x) are no better, since they explicitly use the constant 1 as an additive term and are chosen to nearly minimize the maximum absolute error rather than the relative error. Clenshaw's [3] Chebyshev series expansions for erf $(x)/x$ come close to minimizing relative error, but his approximations are somewhat inefficient because of his choice of interval and his restriction to polynomials.

For a computer subroutine with entries for both erf (x) and erfc (x), cancellation error can be avoided by evaluating erf (x) directly and erfc (x) indirectly (as $1 - \operatorname{erf}(x)$) when erf (x) is smaller in magnitude than erfc (x), and erf (x) indirectly and erfc (x) directly, otherwise. The changeover point occurs for $|x| \simeq .47$.

In this note we present nearly-best rational approximations for the functions erf (x) and erfc (x) with maximal relative errors ranging down to between 6×10^{-19} and 3×10^{-20}. The approximation forms and intervals used are

$$\begin{aligned}
\operatorname{erf}(x) &\simeq x R_{lm}(x^2), & |x| &\leqq .5, \\
\operatorname{erfc}(x) &\simeq e^{-x^2} R_{lm}(x), & .46875 &\leqq x \leqq 4.0, \\
\operatorname{erfc}(x) &\simeq \frac{e^{-x^2}}{x} \left\{ \frac{1}{\sqrt{\pi}} + \frac{1}{x^2} R_{lm}(1/x^2) \right\}, & x &\geqq 4,
\end{aligned}$$

where the $R_{lm}(z)$ are rational functions of degree l in the numerator and m in the denominator. The relations erf $(-x) = -$erf (x) and erfc $(-x) = 2 -$ erfc (x) can be used to evaluate the functions for negative arguments.

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Table I. $\mathcal{E}_{\ell m} = -100 \log_{10} \max \left| \dfrac{f(x) - f_{\ell m}(x)}{f(x)} \right|$

$f(x) = \operatorname{erf}(x), \qquad |x| \le .5$

$m \backslash \ell$	0	1	2	3	4	5	6	7	8
0	139	313	496	688	887	1092			
1	417	556*	753	960	1172	1390			
2	558	702	986*	1212	1438	1666			
3	800	956	1307	1465*	1698	1935			
4	962	1108	1466	1626	1950*				
5	1158	1338	1751	1932					

$f(x) = \operatorname{erfc}(x), \qquad .46875 \le x \le 4.0$

$m \backslash \ell$	0	1	2	3	4	5	6	7	8
0		61	109	161	214	270			
1	164	222*	280	340	401	462			
2		376	441	506	572	638			
3		440	597	666	736	806			
4		502	666	824	897*				
5					1056	1132*			
6						1292	1371		
7							1532	1613*	
8								1775	1859*

$f(x) = \operatorname{erfc}(x), \qquad x \ge 4.0$

$m \backslash \ell$	0	1	2	3	4	5	6	7	8
0			628	756	876				
1		688*	828	958	1081	1198			
2		855	998*	1131	1256				
3		992	1151	1287*	1415				
4		1116	1283	1431	1561*				
5		1232	1405	1558		1824*			

*Coefficients for these approximations only are given in Tables II-IV.

Table II. $\operatorname{erf}(x) \approx f_{nn}(x) = x \sum_{j=0}^{n} p_j x^{2j} \Big/ \sum_{j=0}^{n} q_j x^{2j}$, $|x| \leq .5$

n	j	p_j		q_j	
1	0	3.67678 77	(00)	3.25845 93	(00)
	1	-9.79704 65	(-02)	1.00000 00	(00)
2	0	2.13853 32237 8	(01)	1.89522 57241 5	(01)
	1	1.72227 57703 9	(00)	7.84374 57083 0	(00)
	2	3.16652 89065 8	(-01)	1.00000 00000 0	(00)
3	0	2.42667 95523 05317 5	(02)	2.15058 87586 98612 0	(02)
	1	2.19792 61618 29415 2	(01)	9.11649 05404 51490 1	(01)
	2	6.99638 34886 19135 5	(00)	1.50827 97630 40778 7	(01)
	3	-3.56098 43701 81538 5	(-02)	1.00000 00000 00000 0	(00)
4	0	3.20937 75891 38469 47256 2	(03)	2.84423 68334 39170 62227 3	(03)
	1	3.77485 23768 53020 20813 7	(02)	1.28261 65260 77372 27564 5	(03)
	2	1.13864 15415 10501 55649 5	(02)	2.44024 63793 44441 73305 6	(02)
	3	3.16112 37438 70565 59694 7	(00)	2.36012 90952 34412 09349 9	(01)
	4	1.85777 70618 46031 52673 0	(-01)	1.00000 00000 00000 00000 0	(00)

Table III. $\operatorname{erfc}(x) \approx f_{nn}(x) = e^{-x^2} \sum_{j=0}^{n} p_j x^j \Big/ \sum_{j=0}^{n} q_j x^j$, $\quad .46875 \le x \le 4.0$

n	j	p_j		q_j	
1	0	7.3033	(-01)	6.6211	(-01)
	1	-2.3877	(-02)	1.0000	(00)
4	0	7.37388 83116	(00)	7.37396 08908	(00)
	1	6.86501 84849	(00)	1.51849 08190	(01)
	2	3.03179 93362	(00)	1.27955 29509	(01)
	3	5.63169 61891	(-01)	5.35421 67949	(00)
	4	4.31877 87405	(-05)	1.00000 00000	(00)
5	0	2.28989 92851 659	(01)	2.28989 85749 891	(01)
	1	2.60947 46956 075	(01)	5.19335 70687 552	(01)
	2	1.45718 98596 926	(01)	5.02732 02863 803	(01)
	3	4.26772 01070 898	(00)	2.62887 95758 761	(01)
	4	5.64371 60686 381	(-01)	7.56884 82293 618	(00)
	5	-6.08581 51959 688	(-06)	1.00000 00000 0000	(00)

7	0	3.00459	26102	01616	005	(02)	3.00459	26095	69832	933	(02)
	1	4.51918	95371	18729	422	(02)	7.90950	92532	78980	272	(02)
	2	3.39320	81673	43436	870	(02)	9.31354	09485	06096	211	(02)
	3	1.52989	28504	69404	039	(02)	6.38980	26446	56311	665	(02)
	4	4.31622	27222	05673	530	(01)	2.77585	44474	39876	434	(02)
	5	7.21175	82508	83093	659	(00)	7.70001	52935	22947	295	(01)
	6	5.64195	51747	89739	711	(-01)	1.27827	27319	62942	351	(01)
	7	-1.36864	85738	27167	067	(-07)	1.00000	00000	00000	000	(00)
8	0	1.23033	93547	97997	25272	(03)	1.23033	93548	03749	42043	(03)
	1	2.05107	83778	26071	46532	(03)	3.43936	76741	43721	63696	(03)
	2	1.71204	76126	34070	58314	(03)	4.36261	90901	43247	15820	(03)
	3	8.81952	22124	17690	90411	(02)	3.29079	92357	33459	62678	(03)
	4	2.98635	13819	74001	31132	(02)	1.62138	95745	66690	18874	(03)
	5	6.61191	90637	14162	94775	(01)	5.37181	10186	20098	57509	(02)
	6	8.88314	97943	88375	94118	(00)	1.17693	95089	13124	99305	(02)
	7	5.64188	49698	86700	89180	(-01)	1.57449	26110	70983	47253	(01)
	8	2.15311	53547	44038	46343	(-08)	1.00000	00000	00000	00000	(00)

Table IV. $\operatorname{erfc}(x) \simeq f_{nn}(x) = \frac{e^{-x^2}}{x}\left\{\frac{1}{\sqrt{\pi}} + \frac{1}{x^2} \sum_{j=0}^{n} p_j x^{-2j} \Big/ \sum_{j=0}^{n} q_j x^{-2j}\right\}, \qquad x \geq 4.0$

n	j	p_j		q_j	
1	0	-1.24368 544	(-01)	4.40917 061	(-01)
	1	-9.68210 364	(-02)	1.00000 000	(00)
2	0	-4.25799 64355 3	(-02)	1.50942 07054 5	(-01)
	1	-1.96068 97372 6	(-01)	9.21452 41169 4	(-01)
	2	-5.16882 26218 5	(-02)	1.00000 00000 0	(00)
3	0	-1.21308 27638 9978	(-02)	4.30026 64345 2770	(-02)
	1	-1.19903 95526 8146	(-01)	4.89552 44196 1437	(-01)
	2	-2.43911 02948 8626	(-01)	1.43771 22793 7118	(00)
	3	-3.24319 51927 7746	(-02)	1.00000 00000 0000	(00)
4	0	-2.99610 70770 35421 74	(-03)	1.06209 23052 84679 18	(-02)
	1	-4.94730 91062 32507 34	(-02)	1.91308 92610 78298 41	(-01)
	2	-2.26956 59353 96869 30	(-01)	1.05167 51070 67932 07	(00)
	3	-2.78661 30860 96477 88	(-01)	1.98733 20181 71352 56	(00)
	4	-2.23192 45973 41846 86	(-02)	1.00000 00000 00000 00	(00)
5	0	-6.58749 16152 98378 03157	(-04)	2.33520 49762 68691 85443	(-03)
	1	-1.60837 85148 74227 66278	(-02)	6.05183 41312 44131 91178	(-02)
	2	-1.25781 72611 12292 46204	(-01)	5.27905 10295 14284 12248	(-01)
	3	-3.60344 89994 98044 39429	(-01)	1.87295 28499 23460 47209	(00)
	4	-3.05326 63496 12323 44035	(-01)	2.56852 01922 89822 42072	(00)
	5	-1.63153 87137 30209 78498	(-02)	1.00000 00000 00000 00000	(00)

Table I presents the initial segments of the L_∞ Walsh arrays while Tables II, III, and IV present selected approximations. All approximations were generated using a standard version of the Remes algorithm [4] on a CDC 3600. The master function routines used continued-fraction expansions described in [1] and were verified to be accurate to at least 22S. Finally, the accuracy of the approximations as presented here was verified by comparison against the master routines using 5000 pseudo-random arguments.

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