

Calculation of Dirichlet L -Functions

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Abstract. A method for calculating Dirichlet L -series is presented along with the theory of residue class characters and their automatic generation. Tables are given of zeros of L -series for moduli ≤ 24 .

1. Introduction. In this paper, there is introduced a method for calculating the values of Dirichlet L -functions. The fundamental theorems on characters mod k are given in Section 2, and a numbering of these characters is defined. Formulas are found for the numbers of real and complex characters, both primitive and imprimitive. The automatic generation of a character with a given number is described. In Section 3, the method of asymptotic evaluation is discussed, and a description is given of a calculation of complex zeros of L -functions mod $k \leq 24$ for $|t| \leq 25$. A microfiche at the end of this issue gives these zeros and various other tables and also the FORTRAN programs.

2. Characters. Some of the material in this section is in the "folklore" of the subject and some material is a refinement of known results.

Let \bar{a} be the residue class of a , where the modulus m will be clear from the context. If $c \in \bar{a}$, then $(c, m) = (a, m)$. We write $m = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, where p_1, \dots, p_n are distinct primes. By $M(m)$ we mean the group of residues \bar{a} such that $(a, m) = 1$. We define a mapping $f: M(m) \rightarrow M(p_1^{\alpha_1}) \times \cdots \times M(p_n^{\alpha_n})$, where \times means the usual Cartesian product, by $f(\bar{a}) = (\bar{a}_1, \dots, \bar{a}_n)$, where $a \equiv a_j \pmod{p_j^{\alpha_j}}$. It is easy to see that this is a well-defined map, and, using the Chinese Remainder Theorem, that f is a multiplicative isomorphism of $M(m)$ and the group $M(p_1^{\alpha_1}) \times \cdots \times M(p_n^{\alpha_n})$. Next, for $i = 1, \dots, n$, we define the map $f_i: M(p_i^{\alpha_i}) \rightarrow M(m)$, by $f_i(\bar{a}) = \bar{b}$ where $b \equiv a \pmod{p_i^{\alpha_i}}$, $b \equiv 1 \pmod{p_j^{\alpha_j}}$, $j = 1, \dots, n, j \neq i$. One can easily verify that each of these maps is well-defined and is an into multiplicative isomorphism. Finally, we define the map $h: M(p_1^{\alpha_1}) \times \cdots \times M(p_n^{\alpha_n}) \rightarrow M(m)$ by

$$(1) \quad h((\bar{a}_1, \dots, \bar{a}_n)) = f_1(\bar{a}_1) \cdots f_n(\bar{a}_n),$$

and easily verify that $h(f(\bar{a})) = \bar{a}$.

A short calculation shows that if $i \neq j$, then the only common image of f_i and f_j is $\bar{1}$. Thus, the images under f_1, \dots, f_n of $M(p_1^{\alpha_1}), \dots, M(p_n^{\alpha_n})$ are pairwise disjoint except for the common identity. We designate these images, which are obviously subgroups of $M(m)$, respectively by G_1, \dots, G_n . Since every element of $M(m)$ is expressible as a product such as given in (1), and the cardinalities of G_1, \dots, G_n just multiply to the cardinality of $M(m)$, $M(m)$ is the internal direct product of G_1, \dots, G_n .

Now, we write, for $k > 2$,

$$(2) \quad k = 2^\beta p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \quad 2 < p_1 < \cdots < p_r, \quad r = 0 \text{ if } k = 2^\beta,$$

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and wish to determine a basis for $M(k)$ in a fixed unequivocal manner. For p an odd prime, we set $g_1 = g_1(p) =$ the least primitive root mod p . As shown in Landau [13, pp. 79–81], either g_1 or $g_1 + p$ is a primitive root mod p^α , $\alpha \geq 1$, and we set $g = g(p) =$ the least number which is a primitive root for p^α , $\alpha \geq 1$. Thus, for all odd primes p , $g_1 \leq g \leq g_1 + p$. For $p < 3000$, and possibly further, $g = g_1$. For $p = 2$, we set $g = 3$. For $p = 40487$, $g_1 = 5$, but $g = 10$; cf. [18].

Now, following the method of LeVeque [5, Vol. II, pp. 207–210], we can solve for the basis elements of $M(k)$, B_1, \dots, B_R , where $R = r$ for $\beta \leq 1$, $R = r + 1$ for $\beta = 2$, and $R = r + 2$ for $\beta \geq 3$. For $\beta = 0$, we solve, for $i = 1, \dots, r$, $B_i \equiv g(p_i) \pmod{p_i^{\alpha_i}}$ and $B_i \equiv 1 \pmod{p_j^{\alpha_j}}$ for $1 \leq j \leq r, j \neq i$. If $\beta = 1$, we adjoin to these conditions $B_i \equiv 1 \pmod{2}$. For $\beta = 2$, we solve $B_1 \equiv 3 \pmod{4}$, $B_1 \equiv 1 \pmod{p_j^{\alpha_j}}$, $1 \leq j \leq r$, and for $i = 1, \dots, r$, we solve $B_{i+1} \equiv g(p_i) \pmod{p_i^{\alpha_i}}$, and $B_{i+1} \equiv 1 \pmod{4}$, and $B_{i+1} \equiv 1 \pmod{p_j^{\alpha_j}}$, $1 \leq j \leq r, j \neq i$. Finally, for $\beta \geq 3$, we solve $B_1 \equiv -1 \pmod{2^\beta}$ and $B_1 \equiv 1 \pmod{p_j^{\alpha_j}}$, $1 \leq j \leq r$; $B_2 \equiv 5 \pmod{2^\beta}$ and $B_2 \equiv 1 \pmod{p_j^{\alpha_j}}$, $1 \leq j \leq r$; and for $i = 1, \dots, r$, we solve $B_{i+2} \equiv g(p_i) \pmod{p_i^{\alpha_i}}$, $B_{i+2} \equiv 1 \pmod{2^\beta}$ and $B_{i+2} \equiv 1 \pmod{p_j^{\alpha_j}}$, $1 \leq j \leq r, j \neq i$.

A technical procedure to solve such congruences may be found in Uspensky and Heaslet [6, pp. 189–191]. The process requires the computation of $a^{-1} \pmod{m}$, (where $(a, m) = 1$), which is most easily accomplished by expressing $\phi(m)$ in binary and calculating $a^{-1} \equiv a^{\phi(m)-2}$ by repeated squarings of $a \pmod{m}$.

Using the above remarks, the following theorem is easily proved:

THEOREM 1. *If $2 < k = 2^\beta p_1^{\alpha_1} \dots p_r^{\alpha_r}$, and $r = 0$ if $k = 2^\beta$, then $M(k)$ has a basis of R elements, all of even order ≥ 2 , where*

$$\begin{aligned}
 (3) \quad & R = r \quad \text{if } \beta \leq 1, \\
 & R = r + 1 \quad \text{if } \beta = 2, \\
 & R = r + 2 \quad \text{if } \beta \geq 3.
 \end{aligned}$$

In Rotman [7, pp. 63–65] it is shown that any finite abelian group has a basis and that any two bases have the same cardinality.

The least positive k with R basis elements, for $R = 1, 2, \dots$ is given by 3, 8, 24, 120, 840, 9240, 120120, \dots .

Let h_i be the order of B_i . Recall that a character mod k is a nonzero multiplicative function on the residues mod k which is zero at residues not prime to k . It is shown in LeVeque [5, Vol. II, pp. 210–212] that a character is determined by its values at the basis elements, and the value at B_i can only be one of the h_i th roots of unity. We thus obtain

THEOREM 2. *There are exactly $\phi(k)$ characters mod k .*

Next we show:

THEOREM 3. *For $k > 2$, there are exactly 2^R real characters mod k , where R is the number of basis elements of $M(k)$.*

Proof. A character is real if and only if it is real at the basis elements. For each basis element, exactly two choices of the h_i th root of unity will be real, as h_i is even and ≥ 2 .

Landau [14, p. 414] has shown that all characters mod k are real if and only if $k|24$.

Let $k > 2$ and let χ be a character mod k . We define β_1, \dots, β_R by

$$(4) \quad \chi(B_j) = \exp(2\pi i\beta_j/h_j), \quad 0 \leq \beta_j < h_j; \quad j = 1, \dots, R.$$

Clearly, the R -tuple of nonnegative integers β_j determines and is determined by the character χ . Using this representation, it is now easy to see that the characters form a group isomorphic to $M(k)$ under the mapping $\chi \rightarrow B_1^{\beta_1} \cdot \dots \cdot B_R^{\beta_R}$, (where $(\chi_1 \cdot \chi_2)(a)$ is defined as $\chi_1(a) \cdot \chi_2(a)$).

We now number the characters by defining

$$(5) \quad N = N(\chi) = \beta_1 + \beta_2 h_1 + \beta_3 h_1 h_2 + \dots + \beta_R h_1 h_2 \dots h_{R-1}.$$

It is clear that this is a Cantor numbering system and that the $\phi(k)$ characters will be numbered sequentially from 0 to $\phi(k) - 1$. We designate these characters by χ_0, χ_1, \dots . The character corresponding to our χ_0 is the usual principal χ_0 .

The characters corresponding to the R -tuples $(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, \dots, 0, 1)$ form a basis, and the corresponding N 's are $1, h_1, h_1 \cdot h_2, \dots, h_1 \cdot h_2 \dots h_{R-1}$. It is easy to see from the basis representation that if $d \not\equiv 1 \pmod k$ and $(d, k) = 1$ then some character exists for which $\chi(\bar{d}) \neq 1$. It is clear that the characters can be considered as being defined over the integers.

Using the numbering defined, we can now introduce an unequivocal notation for the L -functions. If the modulus is fixed by the context, we can use $L(S, \chi_N)$. If not, we can use a new notation: $L(S, k, N) = L(S, \chi)$ where χ is character number $N \pmod k$.

To find the number N^* of the conjugate character with N given by (5), we set

$$\begin{aligned} \beta_j^* &= 0, & \text{if } \beta_j &= 0, \\ &= h_j - \beta_j, & \text{if } \beta_j &\neq 0, \end{aligned}$$

and use the β_j^* 's in (5) to form N^* . For a prime p , conjugate pairs are simply N and $p - N - 1$, for $N \geq 1$.

Now we take up the important notion of primitivity. We say a character $\chi \pmod k$ is *imprimitive* if there is a proper divisor K of k such that if $a \equiv b \pmod K$ and $(a, k) = (b, k) = 1$ then $\chi(a) = \chi(b)$; otherwise, the character is called *primitive*. Such a number K is called a modulus of imprimitivity. The principal character for $k > 1$ is imprimitive, taking $K = 1$. Also, 1 is not a modulus of imprimitivity for nonprincipal characters. The study of the number of imprimitive characters is much simplified if we introduce the following notions. Let $m = p_1^{\alpha_1} \dots p_n^{\alpha_n}$, and $f_1, \dots, f_n, G_1, \dots, G_n$ have the same meaning as above. Let $\chi^{(j)}$ be a character $\pmod{p_j^{\alpha_j}}$, $j = 1, \dots, n$. We define, for $\bar{a} = \bar{e}_1 \cdot \dots \cdot \bar{e}_n$, with $\bar{e}_j \in G_j$,

$$(6) \quad \begin{aligned} \chi(\bar{a}) &= \chi^{(1)}(f_1^{-1}(\bar{e}_1)) \cdot \chi^{(2)}(f_2^{-1}(\bar{e}_2)) \cdot \dots \cdot \chi^{(n)}(f_n^{-1}(\bar{e}_n)), \\ \chi(\bar{a}) &= 0 \quad \text{if } (a, m) > 1, \end{aligned}$$

and call the resulting function the *exterior product* of $\chi^{(1)}, \dots, \chi^{(n)}$.

THEOREM 4. *The exterior product of $\chi^{(1)}, \dots, \chi^{(n)}$ is a character $\pmod m$. Every character $\pmod m$ can be written uniquely as such an exterior product.*

Proof. By definition, $\chi(\bar{a}) = 0$ for $(a, m) > 1$. Since for $\bar{a} = 1, \bar{e}_1 = \bar{e}_2 = \dots = \bar{e}_n = \bar{1}$, and $f_j(\bar{1}) = \bar{1}$ for $j = 1, \dots, n$, we have $\chi(\bar{1}) = 1 \neq 0$. It remains to show that χ is multiplicative. For elements in $M(m)$, this follows from the multiplicativeness of $\chi^{(j)}$ and the isomorphic mapping property of f_j , for $j = 1, \dots, n$. The other

cases are trivial. Finally, given a character $\chi \pmod m$, define $\chi^{(j)}(\bar{a}) = \chi(f_j(\bar{a}))$ for $(a, p_j) = 1$, and $\chi^{(j)}(\bar{a}) = 0$ for $(a, p_j) > 1$, for $j = 1, \dots, n$. It is easily seen that $\chi^{(j)}$ is a character mod $p_j^{\alpha_j}$. Also, if $\bar{a} = \bar{e}_1 \cdot \dots \cdot \bar{e}_n$ with $\bar{e}_j \in G_j$,

$$\begin{aligned} \chi(\bar{a}) &= \chi(\bar{e}_1 \cdot \dots \cdot \bar{e}_n) = \chi(\bar{e}_1) \cdot \dots \cdot \chi(\bar{e}_n) \\ &= \chi(f_1(f_1^{-1}(\bar{e}_1))) \cdot \dots \cdot \chi(f_n(f_n^{-1}(\bar{e}_n))) \\ &= \chi^{(1)}(f_1^{-1}(\bar{e}_1)) \cdot \dots \cdot \chi^{(n)}(f_n^{-1}(\bar{e}_n)), \end{aligned}$$

which is the value of the exterior product of $\chi^{(1)}, \dots, \chi^{(n)}$. Since the values of χ are determined at all elements of $M(m)$ by its values on $\bigcup G_j$, two distinct characters mod m must induce for some j two distinct $\chi^{(j)}$'s. Since the image under f_j of $M(p_j^{\alpha_j})$ is G_j , two distinct characters mod $p_j^{\alpha_j}$ will give rise to distinct exterior products.

THEOREM 5. *The exterior product is real if and only if the factors are real.*

Proof. Clear, since the values of χ are determined by the values on $\bigcup G_j$.

THEOREM 6. *The exterior product is primitive if and only if all the factors are primitive.*

Proof. If one of the factors, say $\chi^{(i)}$, is imprimitive, let K be the proper divisor. Let $K' = mK/p_j^{\alpha_j}$ and let $a \equiv b \pmod{K'}$ and $(a, m) = (b, m) = 1$. Then $a \equiv b \pmod{K}$ and $a \equiv b \pmod{p_i^{\alpha_i}}$ for $i \neq j$. Thus, $\chi^{(i)}(a) = \chi^{(i)}(b)$ for $i = j$ or not, so $\chi(a) = \chi(b)$ and K' is a modulus of imprimitivity for χ .

Now let χ have the modulus of imprimitivity $K < m$. Some prime, say p_j , appears in K to the α th power where $\alpha < \alpha_j$. Let $a \equiv b \pmod{p_j^\alpha}$ and $(a, p_j) = (b, p_j) = 1$. Let A be in $f_j(\bar{a})$ and B in $f_j(\bar{b})$. Then, $A \equiv a \pmod{p_j^{\alpha_j}}$, $B \equiv b \pmod{p_j^{\alpha_j}}$ and $A \equiv B \equiv 1 \pmod{p_i^{\alpha_i}}$, $i \neq j$. Hence $A \equiv B \pmod{K}$ so $\chi(A) = \chi(B)$. Thus $\chi^{(j)}(a) = \chi^{(j)}(b)$ and $\chi^{(j)}$ is imprimitive.

COROLLARY. *The number of primitive characters mod m is a multiplicative function of m .*

A slight extension of the arguments above can be used to show

THEOREM 7. *If χ is imprimitive, then there is a least modulus of imprimitivity, and all proper divisors of m which are multiples of this least modulus are also moduli of imprimitivity.*

For a complete and simple development of the theory of moduli of imprimitivity, see Spira [17].

We now count the primitive characters.

THEOREM 8. *If p is a prime, then the number of primitive characters mod p^α is $p - 2$ if $\alpha = 1$ and $p^{\alpha-2}(p - 1)^2$ if $\alpha \geq 2$.*

Proof. The principal character is always imprimitive mod p^α , with modulus of imprimitivity = 1. If $\alpha = 1$, this is the only possible modulus, so the other characters are primitive as they are not identically 1. Since $\phi(p) = p - 1$, there are $p - 2$ primitive characters mod p . Now let $p > 2$ and g and χ_j be as defined above, so $\chi_j(g) = \exp(2\pi ij/\phi(p^\alpha))$ and $0 \leq j < p^{\alpha-1}(p - 1)$. First let $j = sp$. For $s = 0$, we have the principal character, which is imprimitive. Thus, we take $1 \leq s < p^{\alpha-2}(p - 1)$. Let $n_1 \equiv n_2 \pmod{p^{\alpha-1}}$ and $n_1 \not\equiv 0 \pmod{p}$. Let $n_1 \equiv g^u \pmod{p^\alpha}$ and $n_2 \equiv g^v \pmod{p^\alpha}$. Then, $g^u \equiv g^v \pmod{p^{\alpha-1}}$. Since g is a primitive root mod $p^{\alpha-1}$, $u \equiv v \pmod{p^{\alpha-2}(p - 1)}$, or $u = v + cp^{\alpha-2}(p - 1)$. Thus,

$$\begin{aligned} \chi_j(n_1) &= \chi_j(g^u) = \exp (u2\pi isp/(p^{\alpha-1}(p - 1))) \\ &= \exp [(v + cp^{\alpha-2}(p - 1))2\pi is/(p^{\alpha-2}(p - 1))] \\ &= \chi_j(g^v) \cdot \exp (2\pi isc) = \chi_j(g^v) = \chi_j(n_2) . \end{aligned}$$

Thus, χ_j is imprimitive for $j = sp$.

Let now $(j, p) = 1$. Suppose p^γ were a modulus of imprimitivity for χ_j with $\gamma \geq 1$. Let $a \equiv b \pmod{p^\beta}$ and $a \not\equiv 0 \not\equiv b \pmod{p}$. Then $a \equiv b \pmod{p}$. Let $a \equiv g^u \pmod{p^\alpha}$ and $b \equiv g^v \pmod{p^\alpha}$. Since $\chi_j(a) = \chi_j(b)$, we have

$$\exp (2\pi iju/(p^{\alpha-1}(p - 1))) = \exp (2\pi iju/(p^{\alpha-1}(p - 1))),$$

so $uj = vj + 2\pi it$ for some t . Since $(p, j) = 1$, we have $u \equiv v \pmod{p^{\alpha-1}}$. Also $u \equiv v \pmod{p - 1}$ as $g^u \equiv g^v \pmod{p}$. Thus $u \equiv v \pmod{p^{\alpha-1}(p - 1)}$, so $a \equiv b \pmod{p^2}$. Thus χ_j is primitive, and indeed assumes distinct values on the residue classes mod p^α which have elements congruent to a single residue mod p .

Note that in this case of $p > 2$, for $\alpha = 1$, the real nonprincipal character is primitive, but for $\alpha > 1$, the real nonprincipal character is obtained for $j = p^{\alpha-1}(p - 1)/2$, and is hence imprimitive.

Finally, let $p = 2$. For $\alpha = 2$, the result is obtained by calculation, and in this case, the real nonprincipal character is also primitive. Let $\alpha \geq 3$. Then $M(2^\alpha)$ has a basis $-\bar{1}, \bar{5}$, of respective orders 2 and $2^{\alpha-2}$. Thus, if χ is a character mod 2^α , then $\chi(\bar{5}) = \exp (2\pi it/2^{\alpha-1})$, where $0 \leq t < 2^{\alpha-2}$. We will show χ is primitive if and only if t is odd. Let $t = 2s$. If $s = 0$, the character is principal and hence imprimitive. Now let $n_1 \equiv n_2 \pmod{2^{\alpha-1}}$ and let $n_1 \equiv \epsilon_1 5^u \pmod{2^\alpha}$, $n_2 \equiv \epsilon_2 5^v \pmod{2^\alpha}$, where ϵ_1 and ϵ_2 are real and of absolute value 1. Hence $\epsilon_1 \epsilon_2 5^{u-v} \equiv 1 \pmod{2^{\alpha-1}}$, so $\epsilon_1 \epsilon_2 = 1$, as the powers of 5 do not represent $-1 \pmod{2^{\alpha-1}}$ if $\alpha \geq 3$. Thus, $u \equiv v \pmod{2^{\alpha-3}}$. Now

$$\begin{aligned} \chi(n_1) &= \chi(\epsilon_1 5^u) = \chi(\epsilon_1) \exp (2\pi i2su/2^{\alpha-1}) \\ &= \chi(\epsilon_2) \exp (2\pi i(2s)(v + b2^{\alpha-3})/2^{\alpha-2}) \\ &= \chi(\epsilon_2)\chi(5^v) \exp (2\pi i(2s) \cdot b) = \chi(n_2) , \end{aligned}$$

so indeed χ is imprimitive. A similar argument to the one above for $p > 2$ shows that if $j = 2s + 1$, then χ is primitive.

COROLLARY 1. *If $k \equiv 2 \pmod{4}$, then there are no primitive characters mod k .*

COROLLARY 2. *The number of primitive characters mod k is $\sum_{d|k} \mu(n)\phi(n/d)$.*

Using the facts given in the proof above, it is easy to machine generate the N 's corresponding to primitive characters.

For the primitive character mod 2^α to be real, we need $2\pi i(2s + 1)/2^{\alpha-2} = l\pi i$, or $l \cdot 2^{\alpha-3} = 2s + 1$. Thus $\alpha \leq 3$, and these cases are easily settled. From the above remarks on real primitive characters, and from Theorems 5 and 6, we obtain

THEOREM 9. *The number of real primitive characters mod $2^\beta \cdot t$, where $(t, 2) = 1$, is 0 if $\beta = 1$ or $\beta > 3$ or t not squarefree, 1 if $\beta = 0$ or 2 and t squarefree and 2 if $\beta = 3$ and t squarefree.*

The numbers N for real primitive characters are given by

$$N = \frac{1}{2}(h_1 + h_1 h_2 + \dots + h_1 h_2 \dots h_R)$$

if $\beta = 0$ or 2 , t squarefree, and this N and $N - 1$ if $\beta = 3$ and t squarefree. The notation of Rosser [4] of putting a star after the modulus when $\beta = 3$ refers to character N , and the unstarred modulus refers to character $N - 1$. Either of these may have $\chi(-1) = +1$ or -1 .

For machine notation, the values of a character $\chi \pmod k$ can be represented by integers. If $\chi(n) = 0$, we use 0 ; if $\chi(n) = \exp(2\pi it/\phi(k))$, we use t , with $1 \leq t \leq \phi(k)$. In the calculation of a character χ_N , the basis is first determined by the solution of linear congruences, using an internal table of primitive roots mod p^α , or a generator for such primitive roots. The parameter N is then decoded into the β_j 's and the character is computed, using the orders of the basis elements as parameters in the loops. It is convenient to separate out the translation of an R -tuple of exponents of the basis elements into the corresponding residue mod k . Testing is best done by generating the $\phi(k)$ characters, checking that they are distinct, and testing each character to see if it is multiplicative and not identically 0 . Primitivity is also easily checked.

N. G. Čudakov [9] has given a development of the theory of characters based on different methods.

3. An Asymptotic Formula for $L(s, \chi)$. Davies and Haselgrove [1] give an asymptotic formula for $L(s, \chi)$ which requires the computation of coefficients. We give a different formula which merely requires the character, but which has the disadvantage that it loses accuracy near $s = 1$.

The computing times for the two methods are roughly equal. L. Schoenfeld [12] has generalized the Davies-Haselgrove method. Another formula is given in Davies [2].

We have, for $\sigma > 1$

$$(7) \quad L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s} = k^{-s} \sum_{j=1}^k \chi(j) \left[\sum_{n=0}^{\infty} (n + j/k)^{-s} \right],$$

since there is absolute convergence for $\sigma > 1$. By a slight change in the proof of the Euler-Maclaurin formula for $\zeta(s)$, we have

$$(8) \quad \sum_{n=0}^{\infty} (n + j/k)^{-s} = \sum_{n=0}^{N-1} (n + j/k)^{-s} + \frac{1}{2} (N + j/k)^{-s} + (N + j/k)^{1-s}/(s - 1) + \sum_{\nu=1}^m [B_{2\nu}/(2\nu)!] \left(\prod_{r=0}^{2\nu-2} (s + r) \right) (N + j/k)^{1-s-2\nu} + \text{error},$$

and this expression provides an analytic continuation. Combining (7) and (8), we obtain

$$(9) \quad L(s, \chi) = \sum_{j=1}^k \chi(j) \left\{ \sum_{n=0}^{N-1} (kn + j)^{-s} + \frac{1}{2} (kN + j)^{-s} + (kN + j)^{1-s}/(k(s - 1)) + \sum_{\nu=1}^m [B_{2\nu}/(2\nu)!] \left(\prod_{r=0}^{2\nu-2} (s + r) \right) (kN + j)^{1-s-2\nu} \cdot k^{2\nu-1} \right\} + \text{error}.$$

The advantage of programming the first sum from 1 to k is that the routine can be tested using check values of $\zeta(s)$ by setting $\chi(j) = 1$ for $j = 1, \dots, k$. Calculating

formulas very similar to those in Spira [10] can be used. A formula for $L'(s, \chi)$ similar to one in Spira [10] can also be easily found. For values near $s = 1$, one could use the functional equation:

$$(10) \quad \begin{aligned} L(s, \chi) &= 2^s k^{-s} \epsilon \pi^{s-1} \Gamma(1-s) \frac{\sin}{\cos} \left(\frac{1}{2} \pi s \right) L(1-s, \bar{\chi}), \\ \chi(-1) &= \begin{matrix} +1 \\ -1 \end{matrix}, \quad \chi \text{ primitive,} \end{aligned}$$

where

$$(11) \quad \epsilon = \sum_{a=1}^k \chi(a) \frac{\cos}{\sin} \left(\frac{2\pi a}{k} \right), \quad \chi(-1) = \begin{matrix} +1 \\ -1 \end{matrix}.$$

For characters with $\chi(-1) = 1$, we have $L(0, \chi) = 0$. For real primitive characters, with $\chi(-1) = -1$, we have $L(0, \chi) = h$, the class number, for $k > 4$. This follows from the functional equation (10), the class number formula (Davenport [11, pp. 37-51]), and the fact that $\epsilon = \sqrt{k}$ (Landau [13, p. 174, Satz 215]). For $k = 3$ and 4 we obtain 1/3 and 1/2 respectively at 0.

The keyhole integral of Davies-Haselgrove [1, (2.1)] provides a continuation of $L(s, \chi)$ to the entire plane. Putting $s = 0$ in that formula, it follows from Schoenfeld [12], that for any primitive $\chi \pmod{k}$

$$(12) \quad L(0, \chi) = (-1/k) \sum_{j=1}^{k-1} j\chi(j),$$

and using this equation, further check values can be obtained, e.g., mod 13, $L(0, \chi_1) = 1 + i$.

Zeros were calculated for $k \leq 24$ and $|t| \leq 25$. An integration was performed to verify the number of zeros obtained, and the number of sign changes of $Z(t, \chi)$ was also counted (Davies-Haselgrove [1]). All zeros were on $\sigma = 1/2$.

A comparison was made with the zeros calculated by Davies and Haselgrove in [1] and in the manuscript table [16]. The Davies-Haselgrove numbering of characters agrees with the numbering introduced above.

The character opposite Z_{15} in [1, p. 127] is imprimitive with resolving modulus $K = 5$. However, the corresponding table in [16] for the signed modulus of the L -function along $\frac{1}{2} + it$ has changes of sign near the true zeros.

In the [16] tables of zeros of L -series mod 5, 7, 11 and 19, and of the real primitive character L -series (given as factors of Dedekind zeta functions) mod 3, 4, 5 and 20, the following errors were found:

Mod 5. Character 3. $|L'|$ for their zero number 4 should have terminal digits 642.

Mod 11. Character 5. Zero number 1 should be $\frac{1}{2} + i 2.477244$, with $|L'| = 1.41292$. They have erroneously inserted the first zero from Character 6.

Mod 11. Character 6. The numbering of the zeros is off by 1. The first zero should be $\frac{1}{2} + i 2.696004$ with $|L'| = 1.34773$.

Mod 19. Character 1. Missed first zero, $\frac{1}{2} + i 2.392764$ with $|L'| = 1.98624$. Numbering of zeros off by 1.

Mod 19. Character 9. Missed first zero, $\frac{1}{2} + i 1.516084$ with $|L'| = 1.35929$. Numbering of zeros off by 1.

$\zeta \cdot L_{20}$. Zero 16 should be $\frac{1}{2} + i 24.90661$.

There were also numerous one and two unit terminal digit errors. The introduction does not describe some of the tables, and the tables of zeros for the real primitive character L -functions mod 5 and mod 7 were missing. The reproduction has some unreadable pages and some duplicate pages.

The paper [1] contains many errors, which are described in Schoenfeld [12].

Once having the lower zeros of all L -series mod k , it is easy to calculate $E(k) =$ the largest $t > 0$ such that $L(s, \chi) \neq 0$ for $0 < \sigma < 1$, $0 \leq |t| < E(k)$, over all characters χ mod k . As it happens for $k \leq 24$, $k \not\equiv 2 \pmod{4}$, the character for which $L(\sigma + iE(k), \chi) = 0$ is primitive. The zeros missed in the Davies-Haselgrove calculation did not affect the value of $E(k)$.

Shanks and Wrench [15] calculated values of

$$L_a(s) = \sum_{n=1}^{\infty} \left(\frac{-a}{2n+1} \right) (2n+1)^{-s}$$

at integer points. These are indeed L -series, and, for example, in our notation, $L_2(s) = L(s, 8, 3)$.

We now list the tables on the microfiche.

Table I. Basis for $M(k)$, $k = 1(1)200$.

Table II. Characters mod k , $k = 1(1)24$.

Table III. Real and complex primitive character N 's, $k = 1(1)100$.

Table IV. $L(0, \chi)$, $L'(0, \chi)$ for primitive characters mod k , $k \leq 24$, 15D.

Table V. $\text{Re } \epsilon$, $\text{Im } \epsilon$, $\text{Arg } \epsilon$ for primitive characters mod k , $k \leq 24$, 15D.

Table VI. Zeros ρ_n of $L(s, \chi)$ and $L'(\rho_n)$ for $0 < \sigma < 1$, $0 \leq t \leq 25$, for primitive characters mod k , $k \leq 24$, 17S.

Table VII. $E(k)$, $k = 1(1)24$, $k \not\equiv 2 \pmod{4}$, 5D, and N for which attained.

In addition, there is on the microfiche a description and listing of FORTRAN programs for calculating L -functions mod k for $k \leq 2048$.

The calculations were carried out at the Michigan State University Computing Center. Further extensive computations are being carried out on a study of real roots of real L -series (Rosser [3], [4]), and on class numbers of cyclotomic fields. This paper was prepared with partial support from NSF grant GP-8957.

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