Nonnegative Matrix Equations
Having Positive Solutions

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Abstract. Suppose $\bar{A}$ is a nonnegative invertible matrix with a positive diagonal $D = \text{Diag} (\bar{A}) > 0$ and $\bar{y} > 0$ is a positive vector. Let $A = D^{-1}\bar{A}$ and $y = D^{-1}\bar{y}$. If $0 < 2y - Ay$, then $2y - Ay \leq x \leq y$, where $x = A^{-1}y$.

Introduction. The inverse $A^{-1}$ of a given nonnegative invertible matrix, $A$, will usually contain negative elements; and hence for some $y > 0$ the solution vector $x = A^{-1}y$ will have negative components. As suggested in the abstract there is no loss in generality in assuming $\text{Diag} (A) = I$. The condition

\[(1) \quad 0 < 2y - Ay\]

will be shown to imply $0 < x = A^{-1}y$ and to imply that $A$ is diagonally similar to the diagonally dominant matrix $Y^{-1}AY$.

Theorem. Suppose $\bar{A}$ is a nonnegative invertible matrix with a positive diagonal $D = \text{Diag} (\bar{A}) > 0$ and $\bar{y} > 0$ is a positive vector. Let $A = D^{-1}\bar{A}$ and $y = D^{-1}\bar{y}$. If $0 < 2y - Ay$, then $2y - Ay \leq x \leq y$, where $x = A^{-1}y$.

Proof. Let $B = A - I$ then (1) implies $0 < (I - B)y$. We wish to show $2y - Ay \leq x \leq y$, i.e. $(I - B)y \leq (I + B)^{-1}y \leq y$, i.e. $(I - B)y \leq (I - B^2)^{-1} (I - B)y \leq y$. Let $u$ be the positive vector $u = (I - B)y$. We wish to show $u \leq (I - B^2)^{-1} u \leq (I - B)^{-1} u$ which will hold provided $(I - B^2)^{-1}$ and $(I - B)^{-1}$ are nonnegative matrices.

These matrices will be nonnegative provided the corresponding matrix series converge, since

\[I \leq I + B^2 + B^4 + \cdots \leq I + B + B^2 + \cdots \]

implies $I \leq (I - B^2)^{-1} \leq (I - B)^{-1}$.

And the series will converge provided the spectral radius of $B$ satisfies $\rho(B) < 1$. To see that $\rho(B) < 1$, we let $y = Ye$ where $e$ is the vector having all its components equal to 1 and $Y$ is the diagonal matrix corresponding to $y$. Then, $0 < (I - B)y$ implies $Y^{-1}BYe < e$ which implies $\rho(B) = \rho(Y^{-1}BY) < 1$.

Corollary. The inequality $Y^{-1}BYe < e$ also implies that the matrix $(I + Y^{-1}BY) = Y^{-1}(I + B)Y = Y^{-1}AY$ is diagonally dominant.

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