Nonnegative Matrix Equations Having Positive Solutions

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Abstract. Suppose $\tilde{A}$ is a nonnegative invertible matrix with a positive diagonal $D = \text{Diag}(\tilde{A}) > 0$ and $\tilde{y} > 0$ is a positive vector. Let $A = D^{-1}\tilde{A}$ and $y = D^{-1}\tilde{y}$. If $0 < 2y - Ay$, then $2y - Ay \leqq x \leqq y$, where $x = A^{-1}y$.

Introduction. The inverse A^{-1} of a given nonnegative invertible matrix, A, will usually contain negative elements; and hence for some $y > 0$ the solution vector $x = A^{-1}y$ will have negative components. As suggested in the abstract there is no loss in generality in assuming $\text{Diag}(A) = I$. The condition

$$0 < 2y - Ay \tag{1}$$

will be shown to imply $0 < x = A^{-1}y$ and to imply that A is diagonally similar to the diagonally dominant matrix $Y^{-1}AY$.

THEOREM. *Suppose $\tilde{A}$ is a nonnegative invertible matrix with a positive diagonal $D = \text{Diag}(\tilde{A}) > 0$ and $\tilde{y} > 0$ is a positive vector. Let $A = D^{-1}\tilde{A}$ and $y = D^{-1}\tilde{y}$. If $0 < 2y - Ay$, then $2y - Ay \leqq x \leqq y$, where $x = A^{-1}y$.*

Proof. Let $B = A - I$ then (1) implies $0 < (I - B)y$. We wish to show $2y - Ay \leqq x \leqq y$, i.e. $(I - B)y \leqq (I + B)^{-1}y \leqq y$, i.e. $(I - B)y \leqq (I - B^2)^{-1}(I - B)y \leqq y$. Let u be the positive vector $u = (I - B)y$. We wish to show $u \leqq (I - B^2)^{-1}u \leqq (I - B)^{-1}u$ which will hold provided $(I - B^2)^{-1}$ and $(I - B)^{-1}$ are nonnegative matrices.

These matrices will be nonnegative provided the corresponding matrix series converge, since

$$I \leqq I + B^2 + B^4 + \cdots \leqq I + B + B^2 + \cdots \quad \text{implies} \quad I \leqq (I - B^2)^{-1} \leqq (I - B)^{-1}.$$

And the series will converge provided the spectral radius of B satisfies $\rho(B) < 1$. To see that $\rho(B) < 1$, we let $y = Ye$ where e is the vector having all its components equal to 1 and Y is the diagonal matrix corresponding to y. Then, $0 < (I - B)y$ implies $Y^{-1}BYe < e$ which implies $\rho(B) = \rho(Y^{-1}BY) < 1$.

COROLLARY. *The inequality $Y^{-1}BYe < e$ also implies that the matrix $(I + Y^{-1}BY) = Y^{-1}(I + B)Y = Y^{-1}AY$ is diagonally dominant.*

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