Finite-Difference Methods and the Eigenvalue Problem for Nonselfadjoint Sturm-Liouville Operators*

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Abstract. In this paper we analyze the convergence of a centered finite-difference approximation to the nonselfadjoint Sturm-Liouville eigenvalue problem

$$\begin{aligned} &\mathfrak{L}[u] \equiv -[a(x)\,u']' - b(x)u' + c(x)u = \lambda u, \quad 0 < x < 1, \\ &u(0) = u(1) = 0 \end{aligned} \tag{1}$$

where $\mathfrak{L}$ has smooth coefficients and $a(x) \geqq a_0 > 0$ on [0, 1]. We show that the rate of convergence is $O(\Delta x^2)$ as in the selfadjoint case for a scheme of the same accuracy. We also establish discrete analogs of the *Sturm* oscillation and comparison theorems. As a corollary we obtain the result

$$\limsup_{M\to\infty;\ \Delta x\to 0;\ (M+1)\Delta x=1} \left\{ \sum_{p=1}^{M} \frac{\|V^p\|_\infty}{\Lambda_p} \right\} < \infty \tag{2}$$

where $\Delta x = 1/(M+1)$ is the mesh size and Λ_p, V^p are the characteristic pairs of L, the $M \times M$ matrix which approximates $\mathfrak{L}$, and V^p is normalized so that $\|V^p\|_2 = 1$.

1. Introduction. Many authors (e.g. [1], [6], [8], [9]) have studied the convergence of finite-difference methods for selfadjoint Sturm-Liouville eigenvalue problems. In this report we are concerned with the nonselfadjoint problem

$$\begin{aligned} &\mathfrak{L}(u) \equiv -[a(x)u']' - b(x)u' + c(x)u = \lambda u, \qquad 0 < x < 1, \\ &u(0) = u(1) = 0 \end{aligned} \tag{1.1}$$

where $a(x) \geqq a_0 > 0$, $c(x) \geqq 0$, and $b(x)$ are all smooth functions. This problem has an infinite sequence of positive [12, p. 37] and distinct [13, p. 212] eigenvalues

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

and a corresponding sequence of smooth eigenfunctions $u^1(x)$, $u^2(x)$, $u^3(x)$, $\cdots$ which we assume normalized so that

$$\int_0^1 |u^p|^2 dx = 1, \qquad p = 1, 2, \cdots. \tag{1.2}$$

Of course, as is well known, the transformation

$$u(x) = \left[\exp\left(-\frac{1}{2}\int_0^x \frac{b(t)}{a(t)}\,dt\right)\right] v(x) \tag{1.3}$$

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puts (1.1) into the selfadjoint form

$$\hat{L}[v] \equiv -(av')' + (c + \tfrac{1}{2} b' + \tfrac{1}{4}(b^2/a)v = \lambda v , \quad v(0) = v(1) = 0 . \tag{1.4}$$

However, we consider the direct approximation of (1.1) by means of the finite-difference equations

$$-\frac{\{a_{k+1/2}(w_{k+1} - w_k) - a_{k-1/2}(w_k - w_{k-1})\}}{\Delta x^2} - \frac{b_k(w_{k+1} - w_{k-1})}{2\Delta x} + c_k w_k = \Lambda w_k , \qquad k = 1, 2, \cdots, M , \quad w_0 = w_{M+1} = 0 \tag{1.5}$$

where M is a large positive integer, $\Delta x = 1/(M + 1)$ is the mesh spacing and the notation g_k is used for $g(k\,\Delta x)$. Equivalently, we may write (1.5) as the finite-dimensional eigenvalue problem:

$$LW = \Lambda W \tag{1.6}$$

where W is the M component vector

$$W = \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_M \end{bmatrix}$$

and L the $M \times M$ tridiagonal matrix

$$L = \frac{1}{\Delta x^2}\begin{bmatrix} \alpha_1 & \beta_1 & & & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & & \\ \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot \ \cdot & \beta_{M-1} \\ 0 & & & \gamma_M & \alpha_M \end{bmatrix} \tag{1.7}$$

with

$$\alpha_k = [a_{k+1/2} + a_{k-1/2}] + c_k\Delta x^2 \ \beta_k = -[a_{k+1/2} + b_k\Delta x/2] \quad \text{and} \quad \gamma_k = [b_k\Delta x/2 - a_{k-1/2}] \qquad k = 1, 2, \cdots, M . \tag{1.8}$$

We will show that the latter procedure preserves the rate of convergence, namely $O(\Delta x^2)$, *which obtains in the selfadjoint case for a scheme of the same accuracy,* (see [6]). This is Theorem 1.

The matrix L defined above will be shown to be similar to an *oscillation matrix*, by means of a diagonal transformation $\tilde{D}$. Using the basic theorem on oscillation matrices, (see [4], [5]) and the fact that the entries of $\tilde{D}$ alternate in sign, one immediately has a discrete analog of the *Sturm Oscillation Theorem* [13, p. 212, Theorem

2.1] namely L has positive distinct eigenvalues $0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_M$ and if W^j is an eigenvector belonging to Λ_j then W^j has exactly $j - 1$ nodes** in $0 < x < 1$. Moreover the nodes of successive eigenvectors alternate.***

We will also show the following

THEOREM 2. *Let V^j be an eigenvector of L corresponding to Λ_j and let $\delta_{\text{Max}}(V^j)$ be the maximum distance between successive nodes of V^j. Then there exists an integer j_0, independent of M, and a positive constant K_1, such that for $j_0 \leqq j \leqq M$, we have*

$$\delta_{\text{Max}}(V^j) \leqq K_1(\Lambda_j)^{-1/2}. \tag{1.9}$$

In the continuous case, the estimate (1.9) is usually obtained as a corollary to the *Sturm Comparison Theorem* [14, p. 224]. We will base the proof of Theorem 2 on a discrete maximum principle. We remark that in the continuous case, proofs of the oscillation and comparison theorems, based on a maximum principle, have been given by Protter and Weinberger in their recent book. See [12].

2. Symmetrization of the Discrete Problem.

Definitions. For any two M vectors X, Y define their scalar products by

$$\langle X, Y \rangle = \Delta x \sum_{k=1}^{M} x_k \bar{y}_k$$

and let

$$\|X\|_2 = \left\{ \Delta x \sum_{k=1}^{M} |x_k|^2 \right\}^{1/2}$$

be the corresponding norm.

If A is an $M \times M$ matrix then we define

$$\|A\|_2 = \operatorname*{Sup}_{X \neq 0} \frac{\|AX\|_2}{\|X\|_2}.$$

LEMMA 1. *Fix $\Delta x > 0$ sufficiently small. Then there exists a nonsingular, positive, diagonal matrix D such that $D^{-1}LD = \hat{L}$ is a real symmetric matrix. Moreover, $\|D\|_2$, $\|D^{-1}\|_2$ remain bounded as $M \to \infty$, $\Delta x \to 0$, $(M + 1)\Delta x = 1$.*

Proof. We construct such a matrix. Let

$$D = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_M \end{bmatrix} \quad \text{where } d_j \neq 0, j = 1, \cdots, M$$

** As in [5], a "node" of V^j is a point where the graph of V^j, (i.e. the graph of the piecewise-linear function obtained from V^j by linear interpolation) intersects the x-axis.

*** These observations about L are not new. (See Sinden [17] and Varga [18, p. 206].)

and $d_1 = 1$ and let $\hat{L} = D^{-1}LD = (\hat{l}_{ij})$.

Since we require $\hat{L} = \hat{L}^T$ we must have

$$d_i^{-1} l_{ij} d_j = d_j^{-1} l_{ji} d_i \qquad \text{where } L = (l_{ij}) .$$

Further, since $l_{ij} = 0$ for $j > i + 1, j < i - 1$, the d_j's must be determined so that

$$d_i^2 = \frac{l_{i,i-1}}{l_{i-1,i}} d_{i-1}^2 , \qquad i = 2, \cdots, M .$$

Starting from $d_1 = 1$, we may solve recursively to obtain

$$d_i^2 = \prod_{k=1}^{i-1} \left(\frac{\gamma_{k+1}}{\beta_k} \right), \qquad i = 2, \cdots, M$$

and, since $\gamma_k, \beta_k < 0$ for sufficiently small Δx, $d_i^2 > 0$ if Δx is small enough.

With D constructed as above, we have

$$\hat{L} = \frac{1}{\Delta x^2} \begin{bmatrix} \alpha_1 & -(\gamma_2\beta_1)^{1/2} & & & 0 \\ -(\gamma_2\beta_1)^{1/2} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -(\gamma_M\beta_{M-1})^{1/2} \\ 0 & & & -(\gamma_M\beta_{M-1})^{1/2} & \alpha_M \end{bmatrix} \tag{2.1}$$

and we must show that $\|D\|_2$, $\|D^{-1}\|_2$ remain bounded as $M \to \infty$. Let

$$Q_i = \prod_{k=1}^{i-1} \left(1 - \frac{b_{k+1}\Delta x}{2a_{k+1/2}} \right)$$

and

$$P_i = \prod_{k=1}^{i-1} \left(1 + \frac{b_k \Delta x}{2a_{k+1/2}} \right)$$

then $d_i^2 = Q_i/P_i$. Now for sufficiently small Δx,

$$\log \left(1 - \frac{b_{k+1}\Delta x}{2a_{k+1/2}} \right) = - \frac{b_{k+1}\Delta x}{2a_{k+1/2}} + O(\Delta x^2)$$

so that

$$\log Q_i = -\Delta x \sum_{k=1}^{i-1} \frac{b_k}{2a_{k+1/2}} + \Delta x \sum_{k=1}^{i-1} O(\Delta x) .$$

Hence,

$$\lim_{\Delta x\to 0,\, i\to\infty;\; i\Delta x=\bar{x}} [\log Q_i] = -\frac{1}{2}\int_0^{\bar{x}} \frac{b(t)}{a(t)}\,dt\,.$$

Similarly,

$$\lim_{\Delta x\to 0,\, i\to\infty;\; i\Delta x=x} [\log P_i] = \frac{1}{2}\int_0^{\bar{x}} \frac{b(t)}{a(t)}\,dt\,.$$

Consequently,

$$\lim d_i = \left[\exp\left(-\frac{1}{2}\int_0^{\bar{x}} \frac{b(t)}{a(t)}\,dt\right)\right] \leqq K_0 < \infty$$

which shows both $\|D\|_2$, $\|D^{-1}\|_2$ remain bounded as $\Delta x \to 0$, $M \to \infty$, $(M + 1)\Delta x = 1$.

LEMMA 2. *For Δx sufficiently small, the eigenvalues of L are strictly positive and they remain bounded away from zero as $M \to \infty$, $\Delta x \to 0$, $(M + 1)\Delta x = 1$.*

Proof. For Δx sufficiently small, γ_k, $\beta_k < 0$. Hence if $L = (l_{ij})$ and $\Omega_i = \sum_{j\neq i} |l_{ij}|$, then

$$\Omega_i = (a_{i+1/2} + a_{i-1/2})/\Delta x^2$$

and $l_{ii} = (a_{i+1/2} + a_{i-1/2})/\Delta x^2 + c_i \geqq \Omega_i$ since $c_i \geqq 0$.

By Gershgorin's theorem, [7], the eigenvalues of L lie in the union of the discs $|z - l_{ii}| \leqq \Omega_i$ in the complex plane. Hence if Λ is an eigenvalue of L, then $\Lambda \geqq 0$ since Λ is real.

Now let l_h be the finite-difference operator corresponding to $-L$, i.e.

$$[l_h v]_k \equiv -\left[\frac{(a_{k+1/2} + a_{k-1/2}) + c_k\Delta x^2}{\Delta x^2}\right]v_k + \left[\frac{a_{k+1/2} + b_k\Delta x/2}{\Delta x^2}\right]v_{k+1} + \left[\frac{a_{k-1/2} - b_k\Delta x/2}{\Delta x^2}\right]v_{k-1}\,.$$

Then, for sufficiently small Δx, *l_h is of positive type* [3, p. 181] *and so satisfies the discrete maximum principle* [16, p. 23, Lemma 2.3]. Consequently [16, p. 108, Theorem 7.1] if $w(k\Delta x)$, $k = 0, 1, \cdots, M + 1$ is an arbitrary real-valued mesh function, there exists positive constants K and δ such that if $0 < \Delta x < \delta$,

$$\|w\|_\infty \equiv \operatorname*{Max}_k |w_k| \leqq \operatorname{Max}\{|w_0|, |w_{M+1}|\} + K\|(l_h w)\|_\infty\,. \tag{2.2}$$

Now let $V = \{v_k\}_{k=1}^M$ be an eigenvector of L corresponding to Λ. We may assume V to be real. Defining $v_0 = v_{M+1} = 0$, $LV = \Lambda V$ is equivalent to

$$[l_h v]_k = -\Lambda v_k\,, \qquad k = 1, \cdots, M\,. \tag{2.3}$$

Hence, using (2.2) and the fact that $\Lambda \geqq 0$,

$$\|v\|_\infty \leqq K\|(l_h v)\|_\infty = \Lambda K\|v\|_\infty$$

i.e. $\Lambda \geqq 1/K > 0$. Q.E.D.

COROLLARY. *Let Γ be the $M \times M$ matrix given by*

$$\Gamma = \begin{bmatrix} +1 & & & & & 0 \\ & -1 & & & & \\ & & +1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ 0 & & & & & (-1)^{M-1} \end{bmatrix}$$

and let $\hat{L}$ *be defined by* (2.1), *then* $\Gamma^{-1}\hat{L}\Gamma$ *is an oscillation matrix.*

Proof. $\Gamma^{-1}\hat{L}\Gamma$ is a positive-definite real symmetric matrix with positive elements along the first super and sub diagonals. The proof now follows from a theorem of Gantmacher and Krein [4, p. 103].

3. Convergence of the Characteristic Pairs of L. Let $0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_M$ be the eigenvalues of L. Fix a positive integer p and let $V^p(\Delta x)$ be the eigenvector corresponding to $\Lambda_p(\Delta x)$, normalized so that $\|V^p\|_2 = 1$. Let $\tilde{V}^p$ be the continuous piecewise-linear function, vanishing at $x = 0, 1$, and which, in the interior of [0, 1], is obtained from V^p by linear interpolation. Consider the families $\{\Lambda_p(\Delta x)\}$, $\{\tilde{V}^p(\Delta x)\}$ as the mesh size $\Delta x \to 0$.

Using Lemma 1 and considering the symmetrized problem, one can give a direct proof of uniform convergence of $\tilde{V}^p$ to u^p and Λ_p to λ_p as $\Delta x \to 0$. (See [2].) This method of proof is based on the compactness of the family $\{\tilde{V}^p(\Delta x)\}$ in $C[0, 1]$ and has been used several times by Parter (see [9], [10], [11]) but it does not immediately yield estimates on the rates of convergence. Nevertheless we will make use below (see Eq. 3.8 (1)) of the fact that $\Lambda_p \to \lambda_p$ together with Lemma 1 above to obtain these estimates. The proof given below is a modification of that given by Gary in [6] for the selfadjoint case.

THEOREM 1. *Let* Λ_p, V^p *be characteristic pairs of* L *with* $\|V^p\|_2 = 1$. *Let* D *be the diagonal matrix of Lemma* 1. *Let* u^p *be an eigenfunction of* $\mathfrak{L}$ *corresponding to* λ_p *and let* U^p *be the* M *vector obtained from* u^p *by mesh-point evaluation. Assume* $u^p(x)$ *normalized so that*

$$\|D^{-1}U^p\|_2 = \|D^{-1}V^p\|_2 \tag{3.1}$$

then as $\Delta x \to 0$, *we have*

$$|\lambda_p - \Lambda_p| \leqq K\Delta x^2 , \tag{3.2}$$

$$\|U^p - V^p\|_2 \leqq K_1\Delta x^2 \tag{3.3}$$

where K, K_1 *are positive constants depending only on* p.

Proof. Because the difference scheme in (1.5) is properly centered and we assume sufficient smoothness of u^p and the coefficients of $\mathfrak{L}$, we have at the mesh points,

$$\mathfrak{L}[u^p] = LU^p + \tau = \lambda_p U^p \tag{3.4}$$

where τ is the "truncation" error and

$$\|\tau\|_2 \leqq K(p)\,\Delta x^2 \quad \text{where } K \text{ is a constant}. \tag{3.5}$$

Let $\hat{L} = D^{-1}LD$ have orthonormal eigenvectors $X^1, X^2, \cdots, X^M$ and write U^p as a linear combination of the DX^j's:

(3.6) $$U^p = \sum_{j=1}^{M} \sigma_j D X^j$$

so that

$$LU^p = \sum_{j=1}^{M} \sigma_j L D X^j = \sum_{j=1}^{M} \sigma_j \Lambda_j D X^j$$

then

$$\tau = (\lambda_p - L)U^p = \sum_{j=1}^{M} \sigma_j (\lambda_p - \Lambda_j) D X^j$$

and

(3.7) $$\sum_{j=1}^{M} \sigma_j^{\,2} |\lambda_p - \Lambda_j|^2 = \|D^{-1}\tau\|_2^{\,2} \leqq \|D^{-1}\|_2^{\,2} \|\tau\|_2^{\,2} \leqq K_1(p)\Delta x^4$$

where K_1 is a constant.

Now, the eigenvalues of L are distinct and converge to the corresponding distinct eigenvalues of $\mathfrak{L}$. It follows that

(3.8) $$\inf_{j \neq p} \{|\lambda_p - \Lambda_j|\} \geqq \omega_0 > 0$$

for all sufficiently small Δx. Hence, on using (3.7),

(3.9) $$\sum_{j \neq p} \sigma_j^{\,2} \leqq K_1 \Delta x^4 .$$

From (3.9), (3.6) we obtain

(3.10) $$\sigma_p^{\,2} = \|D^{-1}U^p\|_2^{\,2} + O(\Delta x^4) \geqq \omega_1 > 0$$

for all sufficiently small Δx.

Thus

(3.11) $$|\lambda_p - \Lambda_p| \leqq K_2(p)\Delta x^2 .$$

Since $V^p = \beta D X^p$ for some β and $\|X^p\|_2 = 1$ we have

$$|\beta| = \|D^{-1}V^p\|_2 .$$

On taking square roots in (3.10), we have

$$\sigma_p = \|D^{-1}U^p\|_2 + O(\Delta x^4)$$

and we may assume that σ_p and β have the same sign; hence using (3.1),

(3.12) $$(\sigma_p - \beta) = O(\Delta x^4) .$$

Writing $U^p - V^p = \sum_{j \neq p} \sigma_j D X^j + (\sigma_p - \beta) D X^p$ we have

(3.13) $$\|D^{-1}(U^p - V^p)\|_2^{\,2} = \sum_{j \neq p} \sigma_j^{\,2} + (\sigma_p - \beta)^2 = O(\Delta x^4)$$

i.e.

(3.14) $$\|U^p - V^p\|_2^2 \leqq \|D\|_2^2 \|D^{-1}(U^p - V^p)\|_2^2 \leqq K_3(p)\,\Delta x^4. \quad \text{Q.E.D.}$$

Notice that the above inequality also implies uniform convergence at the rate of $O(\Delta x)^{3/2}$.

4. Proof of Theorem 2.

LEMMA 3. *Let* $0 < \Lambda_1 < \cdots < \Lambda_M$ *be the eigenvalues of* L. *Then there exists a positive integer* j_0, *independent of* M, *such that for* $j_0 \leqq j \leqq M$ *we have*

$$K_1 j^2\pi^2 \leqq \Lambda_j \leqq K_2 j^2 \pi^2 , \qquad K_1, K_2 \text{ positive constants} . \tag{4.1}$$

Proof. In the selfadjoint case this result may be found in Bückner [1]. In the present more general case we will need to estimate the off-diagonal elements of the matrix $\hat{L}$ in Lemma 1.

With the notation of (1.8) let

$$q_k^{\,2} = \gamma_{k+1}\beta_k = \left(a_{k+1/2} - \frac{b_{k+1}\Delta x}{2}\right)\left(a_{k+1/2} + \frac{b_k \Delta x}{2}\right), \qquad k = 1, \cdots, M-1 . \tag{4.2}$$

Since $b(x) \in C^1[0, 1]$, we have by the mean-value theorem,

$$q_k^{\,2} = (a_{k+1/2})^2[1 - 2\mu_k \Delta x^2 + O(\Delta x^3)] \tag{4.3}$$

where $2\mu_k = [b_k^2 + 2a_{k+1/2}b'(\xi_k)]/4a_{k+1/2}$ for some ξ_k such that $k\,\Delta x < \xi_k < (k+1)\,\Delta x$. Hence on taking square roots

$$q_k = a_{k+1/2}[1 - \mu_k \Delta x^2 + O(\Delta x^3)] , \qquad k = 1, \cdots, M-1 . \tag{4.4}$$

We now proceed to estimate the quadratic form $\langle X, \hat{L}X\rangle$ where X is any complex M vector of norm 1. Defining $x_0 = x_{M+1} = 0$, and using (4.3), we may write

$$\begin{aligned} \langle X, \hat{L}X\rangle &= \Delta x \sum_{k=0}^{M} \frac{|x_k - x_{k+1}|^2}{\Delta x^2} + \Delta x \sum_{k=1}^{M} c_k |x_k|^2 \\ &\quad + 2\Delta x \sum_{k=0}^{M} \mu_k a_{k+1/2} x_k \bar{x}_{k+1} + O(\Delta x)\Delta x \sum_{k=0}^{M} x_k \bar{x}_{k+1} . \end{aligned} \tag{4.5}$$

Now let $0 < a_0 \leqq a(x) \leqq a_1$ on $[0, 1]$ and let

$$\|c\|_\infty = \operatorname*{Max}_k |c_k| , \qquad \|\mu\|_\infty = \operatorname*{Max}_k |\mu_k| .$$

We have

$$\langle X, \hat{L}X\rangle \leqq a_1 \Delta x \sum_{k=0}^{M} \frac{|x_{k+1} - x_k|^2}{\Delta x^2} + \|c\|_\infty + 2a_1\|\mu\|_\infty + |O(\Delta x)| \tag{4.6}$$

and

$$\langle X, \hat{L}X\rangle \geqq a_0 \Delta x \sum_{k=0}^{M} \frac{|x_{k+1} - x_k|^2}{\Delta x^2} - 2a_1\|\mu\|_\infty - |O(\Delta x)| . \tag{4.7}$$

Let H be the tridiagonal $M \times M$ matrix defined by

$$H = \frac{1}{\Delta x^2}\begin{bmatrix} 2 & & -1 & & 0 \\ & \cdot & & \cdot & \\ -1 & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & \cdot \\ & & \cdot & \cdot & -1 \\ & & \cdot & & \cdot \\ 0 & & -1 & & 2 \end{bmatrix} . \tag{4.8}$$

It is easily verified that

$$\langle X, HX\rangle = \Delta x \sum_{k=0}^{M} \frac{|x_{k+1} - x_k|^2}{\Delta x^2} \tag{4.9}$$

and that the eigenvalues θ_j, $j = 1, \cdots, M$, of H, arranged in increasing order, are given by

$$\theta_j = \frac{4}{\Delta x^2} \sin^2 \frac{j\pi\Delta x}{2}, \qquad j = 1, \cdots, M. \tag{4.10}$$

Inserting (4.9) into (4.6), (4.7) and using the maximum principle for the eigenvalues of real symmetric matrices shows that

$$a_0\theta_j - 2\|\mu\|_\infty - |O(\Delta x)| \leqq \Lambda_j \leqq a_1\theta_j + \|c\|_\infty + 2a_1\|\mu\|_\infty + |O(\Delta x)|. \tag{4.11}$$

Using (4.10) and an elementary calculation, the proof follows from (4.11).

Proof of Theorem 2. Let

$$W^j = \begin{bmatrix} w_1{}^j \\ \cdot \\ \cdot \\ \cdot \\ w_M{}^j \end{bmatrix}$$

be an eigenvector of L corresponding to Λ_j. Then W^j satisfies the difference equations:

$$\begin{aligned} -\left[2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_k}\right] w_k{}^j + \left[\frac{a_{k+1/2} + b_k\Delta x/2}{\omega_k}\right] w_{k+1}^j \\ + \left[\frac{a_{k-1/2} - b_k\Delta x/2}{\omega_k}\right] w_{k-1}^j = 0, \qquad k = 1, \cdots, M \end{aligned} \tag{4.12}$$

where $w_0{}^j = w_{M+1}^j = 0$ and $\omega_k = \frac{1}{2}(a_{k+1/2} + a_{k-1/2})$.

Let

$$\tilde{\alpha}_k = -\left[2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_k}\right], \qquad \tilde{\beta}_k = \left[\frac{a_{k+1/2} + \frac{1}{2} b_k \Delta x}{\omega_k}\right],$$
$$\tilde{\gamma}_k = \left[\frac{a_{k-1/2} - \frac{1}{2} b_k \Delta x}{\omega_k}\right],$$

and let A be the tridiagonal $M \times M$ matrix

$$A = \begin{bmatrix} \tilde{\alpha}_1 & \tilde{\beta}_1 & & & 0 \\ \tilde{\gamma}_2 & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \tilde{\beta}_{M-1} \\ 0 & & & \tilde{\gamma}_M & \tilde{\alpha}_M \end{bmatrix}. \tag{4.13}$$

Then we may write (4.12) as

$$AW^j = 0 \tag{4.14}$$

or equivalently

$$(P^{-1}AP)P^{-1}W^j = 0 \tag{4.15}$$

if P is any nonsingular matrix.

Choose P to be the diagonal matrix

$$P = \begin{bmatrix} p_1 & & & & & 0 \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & p_M \end{bmatrix} \tag{4.16}$$

where $p_1 = 1$ and $p_i^2 = \prod_{k=1}^{i-1} (\tilde{\gamma}_{k+1}/\tilde{\beta}_k)$, $i = 2, \cdots, M$.

For all sufficiently small Δx, $p_i^2 > 0$ and as in Lemma 1, P symmetrizes A. Let $\sigma_k = (\tilde{\gamma}_{k+1}\tilde{\beta}_k)^{1/2}$, then

$$P^{-1}AP = \begin{bmatrix} \tilde{\alpha}_1 & \sigma_1 & & & 0 \\ \sigma_1 & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \sigma_{M-1} \\ 0 & & & \sigma_{M-1} & \tilde{\alpha}_M \end{bmatrix} . \tag{4.17}$$

Observe that by the mean-value theorem

$$\omega_k\omega_{k+1} = (a_{k+1/2})^2[1 + O(\Delta x^2)] \qquad \text{as } \Delta x \to 0 . \tag{4.18}$$

Also if $b(x) \in C^1[0, 1]$,

$$\begin{aligned} (\tilde{\gamma}_{k+1}\tilde{\beta}_k) &= \left[\frac{(a_{k+1/2})^2 + \dfrac{a_{k+1/2}(b_k - b_{k+1})\Delta x}{2} - \dfrac{b_k b_{k+1}\Delta x^2}{4}}{\omega_k\omega_{k+1}}\right] \\ &= \frac{(a_{k+1/2})^2[1 + O(\Delta x^2)]}{(a_{k+1/2})^2[1 + O(\Delta x^2)]} \qquad \text{as } \Delta x \to 0 . \end{aligned} \tag{4.19}$$

Hence,

$$\sigma_k = (\tilde{\gamma}_{k+1}\tilde{\beta}_k)^{1/2} = 1 + O(\Delta x^2) \qquad \text{as } \Delta x \to 0 . \tag{4.20}$$

Let $V = P^{-1}W^j$ and write the system (4.15) as

$$\begin{aligned} -\left[2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_k}\right]v_k + \sigma_k v_{k+1} + \sigma_k v_{k-1} &= 0 , \\ v_0 = v_{M+1} &= 0 , \end{aligned} \qquad k = 1, \cdots, M . \tag{4.21}$$

Let K_1 and K_2 be the constants in Lemma 3 and define

$$\beta_j^2 = \Lambda_j/K_2 . \tag{4.22}$$

Let $y(x) = \sin \beta_j x$. Then $y_k = y(k\,\Delta x)$ satisfies the difference equations:

$$-[2 - \mu_j \Delta x^2] y_k + y_{k+1} + y_{k-1} = 0 , \qquad k = 1, 2, \cdots \tag{4.23}$$

where

$$\mu_j = \frac{4}{\Delta x^2} \sin^2 \frac{\beta_j \Delta x}{2} . \tag{4.24}$$

The distance between successive zeros of $y(x)$ is $\pi/\beta_j = (K_2\pi^2/\Lambda_j)^{1/2} \geqq 1/j$ for j large enough by Lemma 3.

Let $v(x)$ be the piecewise-linear function corresponding to "graph" of vector $V = P^{-1}W^j$. Define the auxiliary function $z(x)$ by

$$z(x) = y(x)/v(x) \qquad \text{whenever } v(x) \neq 0 .$$

We proceed to estimate the distance between successive nodes of $v(x)$ by investigating the difference equation satisfied by $z(x)$.

We may assume that $\delta_{\text{Max}}(V) > 3\,\Delta x$; for if $\delta_{\text{Max}}(V) \leqq 3\,\Delta x$, then in particular, $\delta_{\text{Max}}(V) \leqq 3/(M+1) < 3/j \leqq 3\pi(K_2/\Lambda_j)^{1/2}$ for all sufficiently large j. If $\delta_{\text{Max}} > 3\,\Delta x$, then there exists a set N of consecutive mesh points, containing at least three members on which $v(x)$ is strictly positive (or strictly negative). Let N' be N minus the two end points of N. Since $z_k = y_k/v_k$ for $k \in N'$,

$$\begin{aligned} [l_h z]_k \equiv -\left[\frac{(2 - \mu_j \Delta x^2)\sigma_k}{2 + (c_k - \Lambda_j)\Delta x^2/\omega_k} (v_{k+1} + v_{k-1})\right] z_k & \\ + v_{k+1} z_{k+1} + v_{k-1} z_{k-1} = 0 , \qquad k \in N' . & \end{aligned} \tag{4.25}$$

We now show that for all sufficiently large j, the difference operator l_h (or $-l_h$ if v is strictly negative) *occurring in* (4.25) *is of positive type, and hence satisfies the discrete maximum principle*:

It is sufficient to show that if j is sufficiently large,

$$\frac{[2 - \mu_j \Delta x^2]\sigma_k}{2 + (c_k - \Lambda_j)\Delta x^2/\omega_k} \geqq 1 , \qquad \text{if } k \in N' . \tag{4.26}$$

From (4.24) we have $\mu_j \leqq \Lambda_j/K_2 \leqq \Lambda_j/2a_1$ if K_2 is chosen so that $K_2 \geqq 2a_1$, where a_1 is an upper bound for $a(x)$ on [0, 1]. Hence,

$$(2 - \mu_j \Delta x^2)\sigma_k = 2 - \mu_j \Delta x^2 + O(\Delta x^2) \tag{4.27}$$

since $\mu_j \Delta x^2 \leqq 4$ and $\sigma_k = 1 + O(\Delta x^2)$. Now,

$$\begin{aligned} 2 - \mu_j \Delta x^2 + O(\Delta x^2) &\geqq 2 - \Lambda_j \Delta x^2/K_2 + O(\Delta x^2) \\ &\geqq 2 - \Lambda_j \Delta x^2/2\omega_k + O(\Delta x^2) \\ &= 2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_k} + \frac{(\Lambda_j - 2c_k)\Delta x^2}{2\omega_k} + O(\Delta x^2) \end{aligned}$$

i.e.

$$(2 - \mu_j \Delta x^2)\sigma_k \geqq 2 + (c_k - \Lambda_j)\Delta x^2/\omega_k \tag{4.28}$$

if j is sufficiently large, since we assume $c(x)$ is bounded.

Furthermore, $2 + (c_k - \Lambda_j)\Delta x^2/\omega_k$ is positive for $k \in N'$ since v_k, v_{k+1}, v_{k-1} have the same sign, on using (4.21). Thus (4.26) is satisfied.

Suppose now that $z(x)$ has two zeros in the interval spanned by N. At any mesh point lying between the two zeros we must have $z(x) = 0$ by the maximum principle. Since $z(x) = 0$ if and only if $y(x) = 0$, this means that the distance between successive zeros of $y(x)$ is $\leqq \Delta x = 1/(M + 1)$. However, as already noted, this distance is $\geqq 1/j$ and $j \leqq M$.

Thus $y(x)$ has at most one zero in the interval spanned by N. Hence the maximum distance between successive nodes of $v(x)$ must be less than or equal to $\pi/\beta_j + 2\Delta x$. Since $\Lambda_j = O(1/\Delta x^2)$, we have

$$\delta_{\text{Max}}(V) \leqq K(\Lambda_j)^{-1/2}. \tag{4.29}$$

A similar estimate is valid for the eigenvector W^j of L since $W^j = PV$ and P is a positive diagonal matrix. Q.E.D.

COROLLARY 1. *Let the eigenvectors* $\{V^p\}$ *of* L *be normalized so that* $\|V^p\|_2 = 1$. *Then there exists a constant* K *and an integer* p_0, *both independent of* M *such that if* $p_0 \leqq p \leqq M$

$$\|V^p\|_\infty \equiv \operatorname*{Max}_{k=1\cdots M} |v_k^p| \leqq Kp^{1/2}. \tag{4.30}$$

Proof. Let W^p be the normalized eigenvector of $\hat{L} = D^{-1}LD$ corresponding to Λ_p. Since $W^p = D^{-1}V^p/\|D^{-1}V^p\|_2$ and D^{-1} is a positive diagonal matrix, the distance between successive nodes of W^p satisfies an estimate similar to (4.29). Since W^p is normalized we have

$$\langle W^p, \hat{L}W^p\rangle = \Lambda_p. \tag{4.31}$$

Hence, using inequality (4.7) in the proof of Lemma 3, we get,

$$\Delta x \sum_{k=0}^{M} \frac{|w_{k+1}^p - w_k^p|^2}{\Delta x^2} \leqq \frac{2\Lambda_p}{a_0} \tag{4.32}$$

for all sufficiently large p.

Let r, s be any two positive integers with $1 \leqq s < r \leqq M$. Then,

$$\begin{aligned} |w_r^p - w_s^p| &= \left|\Delta x \sum_{k=s}^{r-1} \frac{w_{k+1}^p - w_k^p}{\Delta x}\right| \\ &\leqq [(r-s)\Delta x]^{1/2}\left(\Delta x \sum_{k=0}^{M} \frac{|w_{k+1}^p - w_k^p|^2}{\Delta x^2}\right)^{1/2} \\ &\leqq [(r-s)\Delta x]^{1/2}(2\Lambda_p/a_0)^{1/2} \end{aligned} \tag{4.33}$$

on using Schwarz's inequality and (4.32). Now choose r so that $|w_r^p| = \|W^p\|_\infty > 0$ and let s be the integer nearest r with the property that $w_s^p w_r^p \leqq 0$. (s need not necessarily be less than r.) We then have for sufficiently large p, by Theorem 2,

$$|(r-s)\Delta x| < 2\delta_{\text{Max}}(W^p) \leqq K'(\Lambda_p)^{-1/2}. \tag{4.34}$$

Hence using (4.33), (4.34)

$$\|W^p\|_\infty \leqq |w_r^p - w_s^p| \leqq [(r-s)\Delta x]^{1/2}(2\Lambda_p/a_0)^{1/2} \leqq K''(\Lambda_p)^{1/4}$$

for sufficiently large p and the proof follows from Lemma 3.

Remark. The estimate (4.30) was obtained by Bückner [1] in the selfadjoint case using an elementary device. It would be interesting to know whether or not the discrete eigenvectors display this growth as $M \to \infty$. In the case of the analytic problem (1.1) it is known (see [15, p. 334]) that the normalized eigenfunctions are uniformly bounded in the supremum norm.

COROLLARY 2. *Let* $\{V^p\}_{p=1}^M$ *be the eigenvectors of L normalized so that* $\|V^p\|_2 = 1$, $p = 1, \cdots, M$. *Then,*

$$\limsup_{M\to\infty;\,\Delta x\to 0;\,(M+1)\Delta x=1} \left\{\sum_{p=1}^{M} \frac{\|V^p\|_\infty}{\Lambda_p}\right\} < \infty .$$

Proof. This follows immediately from Lemmas 2, 3, and Corollary 1.

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1. H. BÜCKNER, "Über Konvergenzsatze, die sich bei der Anwendung eines Differenzenverfahrens auf ein Sturm-Liouvillesches Eigenwertproblem ergeben," *Math. Z.*, v. 51, 1948, pp. 423–465. MR **11**, 58.

2. A. CARASSO, *An Analysis of Numerical Methods for Parabolic Problems Over Long Times*, Ph.D. Thesis, Univ. of Wisconsin, Madison, Wis., 1968.

3. G. E. FORSYTHE & W. R. WASOW, *Finite-Difference Methods for Partial Differential Equations*, Wiley, New York, 1960. MR **23** #B3156.

4. F. R. GANTMACHER, *Matrix Theory*, Vol. II, Chelsea, New York, 1964.

5. F. R. GANTMACHER & M. G. KREIN, *Oscillating Matrices and Kernels and Small Oscillations of Mechanical Systems*, Akademie-Verlag, Berlin, 1960; English transl., U.S.A.E.C. 4481, 1961.

6. J. GARY, "Computing eigenvalues of ordinary differential equations by finite differences," *Math. Comp.*, v. 19, 1965, pp. 365–379. MR **31** #4163.

7. A. S. HOUSEHOLDER, *The Theory of Matrices in Numerical Analysis*, Blaisdell, New York, 1964. MR **30** #5475.

8. H. B. KELLER, "On the accuracy of finite difference approximations to the eigenvalues of differential and integral operators," *Numer. Math.*, v. 7, 1965, pp. 412–419. MR **32** #6706.

9. S. V. PARTER, "Extreme eigenvalues of Toeplitz forms and applications to elliptic difference equations," *Trans. Amer. Math. Soc.*, v. 99, 1961, pp. 153–192. MR **22** #11245.

10. S. V. PARTER, "On the extreme eigenvalues of Toeplitz matrices," *Trans. Amer. Math. Soc.*, v. 100, 1961, pp. 263–276. MR **25** #2421.

11. S. V. PARTER, "On the eigenvalues of certain generalizations of Toeplitz matrices," *Arch. Rational Mech. Anal.*, v. 11, 1962, pp. 244–257. MR **26** #605.

12. M. H. PROTTER & H. F. WEINBERGER, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, N. J., 1967. MR **36** #2935.

13. E. A. CODDINGTON & N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955. MR **16**, 1022.

14. E. L. INCE, *Ordinary Differential Equations*, Dover, New York, 1944. MR **6**, 65.

15. R. COURANT & D. HILBERT, *Methoden der Mathematischen Physik*, Vol. 1, Interscience, New York, 1953. MR **16**, 426.

16. D. GREENSPAN, *Introductory Numerical Analysis of Elliptic Boundary Value Problems*, Harper & Row, New York, 1965. MR **31** #4193.

17. F. W. SINDEN, "An oscillation theorem for algebraic eigenvalue problems and its applications," *Mitt. Inst. Angew. Math. Zürich*, v. 4, 1954, 57 pp. MR **16**, 666.

18. R. S. VARGA, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N. J., 1962. MR **28** #1725.