Table for Third-Degree Spline Interpolation With Equally Spaced Arguments*

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Abstract. A table is given to facilitate the calculation of the parameters of the interpolating third-degree natural spline function for n given data points ($n > 2$) with equally spaced abscissas. The use of the table is described and the correctness of the algorithm is demonstrated.

1. Introduction. Given a set of n real numbers $x_1 < x_2 < \cdots < x_n$ called "knots," a spline function of degree m having the knots x_j is defined to be a function $S(x)$ satisfying the following two conditions:

(1) In each interval (x_j, x_{j+1}) ($j = 0, 1, \cdots, n; x_0 = -\infty, x_{n+1} = \infty$), $S(x)$ is given by some polynomial of degree m (or less).

(2) The polynomial arcs which represent the function in successive intervals join smoothly in the sense that $S(x)$ and its derivatives of order $1, 2, \cdots, m - 1$ are continuous over $(-\infty, \infty)$.

A spline function of odd degree $2k - 1$ is called a "natural" spline function if it satisfies the further condition:

(3) In each of the two intervals $(-\infty, x_1)$ and (x_n, ∞) $S(x)$ is represented by a polynomial of degree $k - 1$ or less (in general, not the same polynomial in the two intervals).

It is well known [1] that given any set of n data points (x_j, y_j) with distinct abscissas, and an integer $k \leqq n$, there is a unique natural spline function $s(x)$ of degree $2k - 1$, having its knots limited to the abscissas x_j, that also interpolates the given data points, in the sense that $s(x_j) = y_j$ ($j = 1, 2, \cdots, n$). Moreover, in the class of continuous functions $f(x)$ with continuous derivatives of order $1, 2, \cdots, k$ on $(-\infty, \infty)$, this natural spline interpolating function is the "smoothest" interpolating function for the given data points, in the sense that the integral

$$\int_a^b [f(x)]^2 dx$$

(for any a, b such that $a \leqq x_1$ and $b \geqq x_n$) is smallest.

Third-degree spline functions (i.e., $k = 2$) have been much more widely used than those of any other degree, and an algorithm is given in [1] for obtaining the third-degree interpolating natural spline function for any set of (2 or more) given data points with distinct abscissas. This algorithm involves the solution of an $(n - 2) \times (n - 2)$ tridiagonal system of linear equations.

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If the abscissas of the data points are equally spaced, substantial simplification is possible, and the parameters of the third-degree interpolating natural spline function can be obtained explicitly, by the use of the table contained in this report, without the necessity of solving a system of equations.

2. Use of the Table. It is assumed that suitable changes of origin and scale have been made, if necessary, so that $x_j = j$ $(j = 1, 2, \cdots, n)$. On this assumption $s(x)$ can be expressed [1] in the form

$$s(x) = s(1) + (x - 1)d + \sum_{j=1}^{n} c_j(x - j)_+^{\,3}\,, \tag{2.1}$$

where the truncated power function z_+^3 is given by

$$\begin{aligned} z_+^{\,3} &= z^3 \qquad (z \geqq 0) \\ &= 0 \qquad (z < 0)\,. \end{aligned}$$

The coefficients d and c_j are to be determined.

TABLE 1

Constants for Calculating Third-Degree Interpolating Natural Spline Function for Equally Spaced Arguments

j	α_j	β_j
2	1	1
3	−6	−4
4	24	15
5	−90	−56
6	336	209
7	−1254	−780
8	4680	2911
9	−17466	−10864
10	65184	40545
11	−2 43270	−1 51316
12	9 07896	5 64719
13	−33 88314	−21 07560
14	126 45360	78 65521
15	−471 93126	−293 54524
16	1761 27144	1095 52575
17	−6573 15450	−4088 55776
18	24531 34656	15258 70529
19	−91552 23174	−56946 26340
20	3 41677 58040	2 12526 34831

The table can be continued by means of the following relations (the first of which does not hold for $j = 3$):

$$\begin{aligned} \alpha_{j+1} &= -4\alpha_j - \alpha_{j-1} \\ \beta_{j+1} &= -4\beta_j - \beta_{j-1} \\ \alpha_j &= \beta_j - 2\beta_{j-1} + \beta_{j-2} \end{aligned}$$

Table 1 gives the values of integer constants α_j and β_j corresponding to each integer $j \geqq 2$. The coefficient d is given by

$$d = [\alpha_2(y_n - y_1) + \alpha_3(y_{n-1} - y_1) + \cdots + \alpha_n(y_2 - y_1)]/\beta_n . \tag{2.2}$$

In order to avoid very rapid accumulation of rounding error (which would otherwise be a serious problem if n is even moderately large), it is suggested that the division by β_n be postponed. Thus d would be retained in the form N/β_n, where N is calculated exactly, using integer or fixed-point arithmetic.

The quantities $\beta_n c_j$ ($j = 1, 2, \cdots, n$) are then obtained recursively by the formulas

$$\beta_n c_1 = \beta_n(y_2 - y_1) - N , \tag{2.3}$$

$$\beta_n c_j = \beta_n(y_{j+1} - y_1) - jN - 2^3\beta_n c_{j-1} - 3^3\beta_n c_{j-2} - \cdots - j^3\beta_n c_1 \quad (j = 2, 3, \cdots, n-1) , \tag{2.4}$$

$$\beta_n c_n = -\beta_n c_1 - \beta_n c_2 - \cdots - \beta_n c_{n-1} , \tag{2.5}$$

again using exact calculation throughout. (The quantities $y_j - y_1$ must, of course, be actually multiplied by β_n.) Finally, N and the quantities $\beta_n c_j$ are divided by β_n to give the parameters d and c_j to the desired precision. It should be borne in mind that in the expression (2.1) the coefficients c_j (especially those with smaller indices) will sometimes be multiplied by large numbers, and may be needed to many decimal places.

3. Derivations and Proofs. Taking $x = k + 1$ in (2.1), transposing certain terms, and noting that $s(k) = y_k$ for $k = 1, 2, \cdots, n$ gives at once

$$c_k = y_{k+1} - y_1 - kd - 2^3c_{k-1} - 3^3c_{k-2} - \cdots - k^3c_1 ,$$

from which (2.4) follows immediately. Similarly, taking $x = 2$ gives (2.3).

Let $\phi(x)$ denote the infinite series

$$\phi(x) = 1^3 + 2^3x + 3^3x^2 + \cdots , \tag{3.1}$$

which converges in the interior of the unit circle. By actual multiplication

$$(1 - x)^4\phi(x) = 1 + 4x + x^2 ,$$

and therefore

$$\varphi(x) = \frac{1 + 4x + x^2}{(1 - x)^4} . \tag{3.2}$$

Further, let

$$\eta(x) = \sum_{j=2}^{\infty} [s(j) - s(1)]x^{j-2} . \tag{3.3}$$

As $s(x)$ is a linear function for $x \geqq n$, this series also converges within the unit circle, as does the binomial expansion

$$(1 - x)^{-2} = 1 + 2x + 3x^2 + \cdots . \tag{3.4}$$

Finally, we denote by $C(x)$ the polynomial

$$C(x) = c_1 + c_2x + \cdots + c_nx^{n-1} . \tag{3.5}$$

From (2.1), (3.1), (3.3), (3.4) and (3.5) we obtain the identity

$$\eta(x) = d(1 - x)^{-2} + \phi(x)C(x) . \tag{3.6}$$

Now, let

$$\psi(x) = \frac{1}{1 + 4x + x^2} . \tag{3.7}$$

Clearly its Maclaurin expansion

$$\psi(x) = \sum_{j=0}^{\infty} b_j x^j = 1 - 4x + 15x^2 - \cdots \tag{3.8}$$

converges in a neighborhood of the origin. Multiplying (3.6) by $(1 - x)^2 \psi(x)$ gives

$$(1 - x)^2 \psi(x) \eta(x) = d\psi(x) + (1 - x)^{-2} C(x) , \tag{3.9}$$

where we have used (3.2) and (3.7). It is shown in [1] that the coefficients c_j satisfy the two conditions

$$c_1 + c_2 + \cdots + c_n = 0 , \tag{3.10}$$

$$c_1 + 2c_2 + \cdots + nc_n = 0 . \tag{3.11}$$

Incidentally, (2.5) follows from (3.10).

Returning, however, to (3.9), we equate coefficients of x^{n-2} on both sides of that equation, noting that the coefficient of x^{n-2} in $(1 - x)^{-2}C(x)$ is

$$\begin{aligned}(n - 1)c_1 &+ (n - 2)c_2 + \cdots + 2c_{n-2} + c_{n-1} \\ &= n(c_1 + c_2 + \cdots + c_n) - (c_1 + 2c_2 + \cdots + nc_n) = 0 ,\end{aligned}$$

by (3.10) and (3.11). Further, let

$$(1 - x)^2 \psi(x) = \sum_{j=0}^{\infty} a_j x^j , \tag{3.12}$$

a series having the same region of convergence as that in (3.8). We obtain, therefore,

$$a_0(y_n - y_1) + a_1(y_{n-1} - y_1) + \cdots + a_{n-2}(y_2 - y_1) = db_{n-2} . \tag{3.13}$$

Finally, we redesignate the coefficients a_j and b_j as α_j and β_j, shifting the indices (for notational convenience in the use of Table 1) so that $\alpha_j = a_{j-2}$ and $\beta_j = b_{j-2}$. Making these substitutions in (3.13) at once gives (2.2). The recurrence relation for the quantities α_j follows from (3.7) and (3.12); that for the β_j from (3.7) and (3.8). The relation $\alpha_j = \beta_j - 2\beta_{j-1} + \beta_{j-2}$ is an immediate consequence of (3.8) and (3.12).

4. Illustrative Example. The values of j and y_j in Table 2, due to K. A. Innanen [2], represent ten points on a segment of a theoretical rotation curve of the galactic system. Here y_j is the circular velocity in the galactic plane in km/sec at a distance of j kiloparsecs from the galactic center. Substituting in (2.2) the values of α_j from Table 1 and those of $y_j - y_1$ from Table 2 gives

$$\begin{aligned} d &= [1(-24.0) - 6(-22.5) + 24(-23.0) - \cdots + 65184(-23.0)]/40545 \\ &= -1005780/40545 = -67052/2703 = -24.8065 . \end{aligned}$$

TABLE 2

Illustrative Data

j	y_j	$y_j - y_1$	$2703c_j$	c_j
1	244.0	0.0	4883.0	1.8065
2	221.0	−23.0	−2268.0	−0.8391
3	208.0	−36.0	−9849.0	−3.6437
4	208.0	−36.0	7876.5	2.9140
5	211.5	−32.5	−2736.0	−1.0122
6	216.0	−28.0	3067.5	1.1349
7	219.0	−25.0	−1425.0	−0.5272
8	221.0	−23.0	−70.5	−0.0261
9	221.5	−22.5	1707.0	0.6315
10	220.0	−24.0	−1185.5	−0.4386

Values of $2703c_j$ are calculated exactly, using (2.3), (2.4), and (2.5). Finally, division by 2703 gives the values of c_j, shown in the last column of Table 2 to four decimal places. Thus, the third-degree interpolating natural spline function for these data is

$$\begin{aligned}244.0 &- 24.8065(x-1) + 1.8065(x-1)_+^3 - 0.8391(x-2)_+^3\\ &- 3.6437(x-3)_+^3 + 2.9140(x-4)_+^3 - 1.0122(x-5)_+^3\\ &+ 1.1349(x-6)_+^3 - 0.5272(x-7)_+^3 - 0.0261(x-8)_+^3\\ &+ 0.6315(x-9)_+^3 - 0.4386(x-10)_+^3 .\end{aligned}$$

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