Elliptic Spline Functions and the Rayleigh-Ritz-Galerkin Method*

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Abstract. Error estimates for the Rayleigh-Ritz-Galerkin method, using finite-dimensional spline type spaces, for a class of nonlinear two-point boundary value problems are discussed. The results of this paper extend and improve recent corresponding results of B. L. Hulme, F. M. Perrin, H. S. Price, and R. S. Varga.

1. Introduction. The purpose of this paper is to discuss error bounds for the Rayleigh-Ritz-Galerkin method, using finite-dimensional "spline-type" spaces, for a class of nonlinear two-point boundary value problems, cf. [2], [3], [4], [5], [7], and [9]. In particular, we generalize, extend, and simplify the very important *techniques* and *results* of [7], [9], and [10].

In Section 2, we generalize the interpolation theory results of [11] and [12] to spline spaces defined by an arbitrary selfadjoint, elliptic, ordinary differential operator and in Section 3 we analyze and apply these results to the class of nonlinear two-point boundary value problems previously studied extensively in [3] and [4]. Now we introduce some notations, which will be used throughout this paper.

Let a and b be two fixed real numbers such that $-\infty < a < b < \infty$. If $u \in C^\infty(a, b)$ and is real-valued, for each nonnegative integer m and $1 \leqq p \leqq \infty$, let

$$\|u\|_{m,p} \equiv \left(\int_a^b \sum_{j=0}^m |D^j u(x)|^p dx\right)^{1/p}, \quad \text{where } D \equiv \frac{d}{dx},$$

$W^{m,p}$ denote the closure of the set $\{u \in C^\infty(a, b), u \text{ real-valued} \mid \|u\|_{m,p} < \infty\}$ with respect to $\|\cdot\|_{m,p}$ and $W_0{}^{m,p}$ denote the closure of the real-valued functions in $C_0{}^\infty(a, b)$, i.e., the real-valued $C^\infty(a, b)$-functions with compact support contained in the interior of (a, b), with respect to $\|\cdot\|_{m,p}$. We remark that $u \in W^{m,p}$ if and only if $u \in C^{m-1}[a, b]$, $D^{m-1}u$ is absolutely continuous, and $D^m u \in L^p[a, b]$. Moreover, $u \in W_0{}^{m,p}$ if and only if $u \in W^{m,p}$ and $D^k u(a) = D^k u(b) = 0$, $0 \leqq k \leqq m - 1$. Finally, the symbol K will be used repeatedly to denote a positive constant, not necessarily the same at each occurrence.

2. γ-Elliptic Spline Functions. In this section, we define the concept of a "γ-elliptic spline space" and we define and analyze a particular interpolation mapping into such a space. In particular, we derive computable lower and upper bounds for the interpolation error. These results generalize those of [11] and [12].

For each nonnegative integer, M, let $\mathcal{P}_M$ denote the set of all partitions, Δ, of the interval $[a, b]$ of the form

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$$(2.1) \qquad \Delta : a = x_0 < x_1 < \cdots < x_M < x_{M+1} = b\,.$$

Moreover, let $\mathcal{P} \equiv \bigcup_{M=0}^{\infty} \mathcal{P}_M$.

Let

$$E(u) \equiv \sum_{j=0}^{m} (-1)^j D^j(p_j(x) D^j u(x)),\, x \in (a, b)\,, \quad \text{where } p_m(x) \geqq \alpha > 0$$

for all $x \in (a, b)$, $p_j(x) \in W^{j,2} \cap W^{0,\infty}$, $0 \leqq j \leqq m$, and E is *γ-elliptic*, i.e., there exists a positive constant, γ, such that

$$\|D^m u\|_{0,2}^2 \leqq \gamma^2 \int_a^b \sum_{j=0}^{m} p_j(x)(D^j u(x))^2 dx \equiv \gamma^2 e(u, u)\,, \quad \text{for all } u \in W_0^{m,2}\,.$$

If $\Delta \in \mathcal{P}_M$ and z is an integer such that $m - 1 \leqq z \leqq 2m - 2$, we define the *$\gamma$-elliptic spline space*, $S(E, \Delta, z)$, to be the set of all real-valued functions $s(x) \in C^z[a, b]$ such that on each subinterval (x_i, x_{i+1}), $0 \leqq i \leqq M$, $E(s(x)) = 0$, for almost all $x \in (x_i, x_{i+1})$.

We remark that in the special case of $p_m(x) = 1$, for all $x \in [a, b]$, and $p_j(x) = 0$, for all $x \in [a, b]$, $0 \leqq j \leqq m - 1$, our definition is identical with the definition of the deficient splines of degree $2m - 1$ of Ahlberg and Nilson, cf. [1]. It is easy to verify that all the results of this paper remain essentially unchanged if one allows the number z to depend on the partition points, x_i, $1 \leqq i \leqq M$, in such a way that $m - 1 \leqq z(x_i) \leqq 2m - 2$ for all $1 \leqq i \leqq M$. The details are left to the reader.

Following [12] we define the interpolation mapping $\mathcal{I}: C^{m-1}[a, b] \rightarrow S(E, \Delta, z)$ by $\mathcal{I}(f) \equiv s$, where

$$(2.2) \qquad D^k s(x_i) \equiv D^k f(x_i)\,, \quad \begin{cases} 0 \leqq k \leqq 2m - 2 - z\,, & 1 \leqq i \leqq M\,, \\ 0 \leqq k \leqq m - 1\,, & i = 0 \text{ and } M + 1\,. \end{cases}$$

We remark that the preceding interpolation mapping corresponds to the Type I interpolation of [12]. It is easy to modify the results of this paper for the cases in which the interpolation mapping corresponds to Types II, III, and IV interpolation of [12]. The details are left to the reader.

We now state and prove analogues of some of the basic results of [12].

THEOREM 2.1. *The interpolation mapping given by* (2.2) *is well defined for all* $\Delta \in P$, *γ-elliptic operators* E, *and* $m - 1 \leqq z \leqq 2m - 2$.

Proof. By [6, p. 43], there exist $2m$ linearly independent functions $v_k(x) \in W^{2m,2}$, $1 \leqq k \leqq 2m$, such that $E(v_k(x)) = 0$ almost everywhere in $[a, b]$, $1 \leqq k \leqq 2m$, and if $s(x)$ is a $S(E, \Delta, z)$-spline function, then on each subinterval (x_i, x_{i+1}), $0 \leqq i \leqq M$, $s(x)$ can be expressed as $s(x) = \sum_{k=1}^{2m} \alpha_{i,k} v_k(x)$. Thus, the total number of coefficients determining $s(x)$ in $[a, b]$ is $2m(M + 1)$.

We now calculate the number of *linear* equations which constrain these coefficients. The regularity conditions, at the interior partition points, yield $(z + 1)M$ linear constraints and the interpolation conditions yield $(2m - 1 - z)M + 2m$ linear constraints. Hence, the total number of linear constraints is $2m(M + 1)$. In other words, if $s(x)$ exists, it is obtained from a solution of $2m(M + 1)$ linear equations in $2m(M + 1)$ unknowns. To establish both the existence and uniqueness of $s(x)$, it suffices to show that if $D^k f(x_i) = 0$, $0 \leqq k \leqq 2m - 2 - z$, $1 \leqq i \leqq M$, and $D^k f(a) = D^k f(b) = 0$, $0 \leqq k \leqq m - 1$, then $s(x) \equiv 0$.

Consider

$$e(s, s) \equiv \int_a^b \sum_{j=0}^m p_j(x)[D^j s(x)]^2 dx = \sum_{i=0}^M \int_{x_i}^{x_{i+1}} p_j(x)[D^j s(x)]^2 dx .$$

Integrating by parts, we have

$$\begin{aligned} e(s, s) &= \sum_{i=0}^M \int_{x_i}^{x_{i+1}} s(x) \sum_{j=0}^m (-1)^j D^j [p_j(x) D^j s(x)] dx \\ &+ \sum_{i=0}^M \sum_{j=0}^m \sum_{k=0}^m (-1)^{j+k} D^{j-k}[p_j(x) D^j s(x)] D^{k-1} s(x) \Big|_{x=x_i}^{x=x_{i+1}} . \end{aligned} \tag{2.3}$$

The first sum in (2.3) is zero since $s(x)$ is a spline function and, using the regularity and interpolation properties of $s(x)$, it may be shown that the second sum is also zero. Hence, $0 = e(s, s) \geqq \gamma \|D^m s\|_{0,2}^2$ and $s(x) \equiv 0$. Q.E.D.

COROLLARY.

$$e(f - \mathcal{S}f, s) = 0 , \tag{2.4}$$

for all $f \in W^{m,2}$, $s(x) \in S(E, \Delta, z)$, E *γ-elliptic,* $\Delta \in \mathcal{P}$, *and* $m - 1 \leqq z \leqq 2m - 2$.

Proof. As in (2.3), we have

$$\begin{aligned} e(f - \mathcal{S}f, s) &= \sum_{i=0}^M \int_{x_i}^{x_{i+1}} (f - \mathcal{S}f) \sum_{j=0}^m (-1)^j D^j [p_j(x) D^j s(x)] dx \\ &+ \sum_{i=0}^M \sum_{j=0}^m \sum_{k=0}^m (-1)^{j+k} D^{j-k}[p_j(x) D^j s(x)] D^k (f - \mathcal{S}f) \Big|_{x=x_i}^{x=x_{i+1}} = 0 . \end{aligned}$$

Q.E.D.

The following result is a generalization of the first integral relation of [12].

THEOREM 2.2. *Let* $f \in W^{m,2}$, *E be a γ-elliptic operator,* $\Delta \in \mathcal{P}$, *and* $m - 1 \leqq z \leqq 2m - 2$. *If* $\mathcal{S}f$ *denotes the* $S(E, \Delta, z)$*-interpolate of* f, *then*

$$e(f, f) = e(f - \mathcal{S}f, f - \mathcal{S}f) + e(\mathcal{S}f, \mathcal{S}f) . \tag{2.5}$$

Proof. Since $e(\cdot, \cdot)$ is a bilinear functional, $e(f, f) = e((f - \mathcal{S}f) + \mathcal{S}f, (f - \mathcal{S}f) + \mathcal{S}f) = e(f - \mathcal{S}f, f - \mathcal{S}f) + 2e(f - \mathcal{S}f, \mathcal{S}f) + e(\mathcal{S}f, \mathcal{S}f)$. The result now follows directly from the above corollary. Q.E.D.

The following result is a generalization of the second integral relation of [12].

THEOREM 2.3. *Let* $f \in W^{2m,2}$, *E be a γ-elliptic operator,* $\Delta \in \mathcal{P}$, *and* $m - 1 \leqq z \leqq 2m - 2$. *If* $\mathcal{S}f$ *denotes the* $S(E, \Delta, z)$*-interpolate of* f, *then*

$$e(f - \mathcal{S}f, f - \mathcal{S}f) = \int_a^b (f - \mathcal{S}f) E(f) dx . \tag{2.6}$$

Proof. As before,

$$\begin{aligned} e(f - \mathcal{S}f, f - \mathcal{S}f) &= \sum_{i=0}^M \int_{x_i}^{x_{i+1}} (f - \mathcal{S}f) E(f - \mathcal{S}f) dx + \sum_{i=0}^M \sum_{j=0}^m \sum_{k=0}^m (-1)^{j+k} \\ &\quad \cdot D^{j-k}[p_j(x) D^j (f(x) - \mathcal{S}f(x))] D^k (f(x) - \mathcal{S}f(x)) \Big|_{x=x_i}^{x=x_{i+1}} \\ &= \int_a^b (f - \mathcal{S}f) E(f - \mathcal{S}f) dx = \int_a^b (f - \mathcal{S}f) E(f) dx , \end{aligned}$$

since $\mathcal{S}f$ is a spline function. Q.E.D.

Finally, following Kolmogorov, cf. [8, p. 146], if t and d are positive integers, let $\lambda_d(t)$ denote the dth eigenvalue of the boundary value problem,

$$(2.7) \qquad (-1)^t D^{2t} y(x) = \lambda y(x) , \qquad a < x < b ,$$

$$(2.8) \qquad D^k y(a) = D^k y(b) = 0 , \qquad t \leqq k \leqq 2t - 1 ,$$

where the λ_d are arranged in order of increasing magnitude and repeated according to their multiplicity. We remark that the problem (2.7) — (2.8) has a countably infinite number of eigenvalues, all of which are nonnegative, and it may be shown that

$$\lambda_d = \left(\frac{\pi}{b-a}\right)^{2t} d^{2t}[1 + O(d^{-1})] , \quad \text{as } t < d \to \infty .$$

Now we obtain *explicit upper* and *lower* bounds for the quantities $\Lambda(E, p, z, j)$, $1 \leqq m$, $p = m$ or $2m$, $m - 1 \leqq z \leqq 2m - 2$, and $0 \leqq j \leqq m$, defined by

$$(2.9) \quad \Lambda(E, m, z, j) \equiv \sup \{ \|D^j(f - \mathcal{S}f)\|_{0,2}/[e(f, f)]^{1/2} | f \in W^{m,2}, e(f, f) \neq 0 \}$$

and

$$(2.10) \quad \Lambda(E, 2m, z, j) \equiv \sup \{ \|D^j(f - \mathcal{S}f)\|_{0,2}/\|E(f)\|_{0,2} | f \in W^{2m,2}, \|E(f)\|_{0,2} \neq 0 \} .$$

Letting

$$\bar{\Delta} \equiv \max_{0 \leqq i \leqq M} (x_{i+1} - x_i) \text{ and } \tilde{\Delta} \equiv \min_{0 \leqq i \leqq M} (x_{i+1} - x_i)$$

for all $\Delta \in \mathcal{P}_M$, we have the following generalization of Theorem 3.4 of [11] and Theorem 7 of [12].

THEOREM 2.4.

$$(2.11) \qquad \lambda_{d+1}^{-1/2}(m - j) \leqq \Lambda(E, m, z, j) \leqq \gamma K_{m,m,z,j} (\bar{\Delta})^{m-j} ,$$

where

$$(2.12) \qquad d \equiv \dim \{ D^j[S(E, \Delta, z)] \}$$

and

$$
\begin{aligned}
K_{m,m,z,j} &= 1 , \quad \text{if } m - 1 \leqq z \leqq 2m - 2, j = m , \\
&= \left(\frac{1}{\pi}\right)^{m-j} , \quad \text{if } m - 1 = z, 0 \leqq j \leqq m - 1 , \\
(2.13) \qquad &= \frac{(z + 2 - m)!}{\pi^{m-j}} , \quad \text{if } m - 1 \leqq z \leqq 2m - 2, 0 \leqq j \leqq 2m - 2 - z , \\
&= \frac{(z + 2 - m)!}{j! \pi^{m-j}} , \quad \text{if } m - 1 \leqq z \leqq 2m - 2, 2m - 2 - z \leqq j \leqq m - 1 ,
\end{aligned}
$$

for all γ-elliptic operators E, $0 \leqq M$, $\Delta \in \mathcal{P}_M$, $m - 1 \leqq z \leqq 2m - 2$, and $0 \leqq j \leqq m$.

Proof. First, we prove the right-hand inequality of (2.11). If $m - 1 \leqq z \leqq 2m - 2$ and $j = m$, the result follows directly from Theorem 2.2.

Otherwise, $D^j(f - \mathcal{S}f)(x_i) = 0$, $1 \leqq i \leqq M$, $0 \leqq j \leqq 2m - 2 - z$, and by the Rayleigh-Ritz inequality, cf. [5, p. 184],

$$\int_{x_i}^{x_{i+1}} (D^j(f - \mathcal{S}f)(x))^2 dx \leqq \left(\frac{\bar{\Delta}}{\pi}\right)^2 \int_{x_i}^{x_{i+1}} (D^{j+1}(f - \mathcal{S}f)(x))^2 dx\,, \quad 0 \leqq j \leqq 2m - 2 - z\,. \tag{2.14}$$

Summing both sides of (2.14) with respect to i from 0 to M, we obtain

$$\|D^j(f - \mathcal{S}f)\|_{0,2} \leqq \frac{\bar{\Delta}}{\pi} \|D^{j+1}(f - \mathcal{S}f)\|_{0,2}\,, \qquad 0 \leqq j \leqq 2m - 2 - z\,. \tag{2.15}$$

Using (2.15) repeatedly, we obtain

$$\|D^j(f - \mathcal{S}f)\|_{0,2} \leqq \left(\frac{\bar{\Delta}}{\pi}\right)^{2m-1-z-j} \|D^{2m-1-z}(f - \mathcal{S}f)\|_{0,2}\,. \tag{2.16}$$

Hence, if $2m - 1 - z = m$, i.e., $z = m - 1$, then

$$\|D^j(f - \mathcal{S}f)\|_{0,2} \leqq \left(\frac{1}{\pi}\right)^{m-j} (\bar{\Delta})^{m-j} \|D^m f\|_{0,2}\,, \tag{2.17}$$

and the required result follows from the γ-ellipticity of E.

Otherwise, since $m \leqq z$, applying Rolle's Theorem to $D^{2m-2-z}(f - \mathcal{S}f) \in C^{z-m+1}[a, b]$, we have that for each $0 \leqq j \leqq z - m + 1$, there exist points $\{\xi_l^{(j)}\}_{l=0}^{m+1-j}$ in $[a, b]$ such that

$$D^{2m-2-z+j}(f - \mathcal{S}f)(\xi_l^{(j)}) = 0\,, \quad 0 \leqq j \leqq m - 1 - (2m - 2 - z) = z - m + 1\,, \quad 0 < l < M + 1 - j\,, \tag{2.18}$$

$$a = \xi_0^{(j)} < \xi_1^{(j)} < \cdots < \xi_{M+1-j}^{(j)} = b\,, \qquad 0 \leqq j \leqq z - m + 1\,, \tag{2.19}$$

$$\xi_l^{(j)} \leqq \xi_l^{(j+1)} < \xi_{l+1}^{(j)}\,, \quad \text{for all } 0 \leqq l \leqq M + 1 - j\,, \quad 0 \leqq j \leqq z - m + 1\,, \tag{2.20}$$

and

$$|\xi_{l+1}^{(j)} - \xi_l^{(j)}| \leqq (j + 1)\bar{\Delta}\,, \qquad 0 \leqq l \leqq M - j\,, \quad 0 \leqq j \leqq z - m + 1\,, \tag{2.21}$$

i.e., choose $\xi_l^{(0)} = x_l$, $0 \leqq l \leqq M + 1$.

Thus, applying the Rayleigh-Ritz inequality, we have

$$\int_{\xi_l^{(j)}}^{\xi_{l+1}^{(j)}} (D^{2m-2-z+j}(f - \mathcal{S}f)(x))^2 dx \leqq \left[\frac{(j + 1)\bar{\Delta}}{\pi}\right]^2 \int_{\xi_l^{(j)}}^{\xi_{l+1}^{(j)}} (D^{2m-2-z+(j+1)}(f - \mathcal{S}f)(x))^2 dx \tag{2.22}$$

for all $0 \leqq l \leqq M - j$, $0 \leqq j \leqq z - m + 1$. Summing (2.22) with respect to l from 0 to $M - j$, we have

$$\|D^{2m-2-z+j}(f - \mathcal{I}f)\|_{0,2} \leqq \frac{(j+1)\bar{\Delta}}{\pi} \|D^{2m-2-z+(j+1)}(f - \mathcal{I}f)\|_{0,2}, \tag{2.23}$$
$$0 \leqq j \leqq z - m + 1 .$$

Using (2.23) repeatedly, we have

$$\begin{aligned} \|D^{2m-1-z}(f - \mathcal{I}f)\|_{0,2} &\leqq \frac{(z+2-m)!}{\pi^{m-(2m-1-z)}} (\bar{\Delta})^{m-(2m-1-z)} \|D^m(f - \mathcal{I}f)\|_{0,2} \\ &\leqq \frac{(z+2-m)!}{\pi^{z-m+1}} (\bar{\Delta})^{z-m+1} \|D^m f\|_{0,2} . \end{aligned} \tag{2.24}$$

Combining (2.16) with (2.24), we have that

$$\|D^j(f - \mathcal{I}f)\|_{0,2} \leqq \frac{(z+2-m)!}{\pi^{m-j}} (\bar{\Delta})^{m-j} \|D^m f\|_{0,2}, \quad \text{if } 0 \leqq j \leqq 2m - 2 - z . \tag{2.25}$$

Otherwise, it follows from (2.23) that

$$\|D^j(f - \mathcal{I}f)\|_{0,2} \leqq \frac{(z+2-m)!}{j!\pi^{m-j}} \|D^m f\|_{0,2} . \tag{2.26}$$

The required result now follows from (2.25), (2.26), and the γ-ellipticity of E.

Finally, we remark that the left-hand inequality of (2.11) follows directly from a fundamental result of Kolmogorov, cf. [8, p. 146]. Q.E.D.

The next result generalizes Theorem 3.5 of [11] and Theorem 9 of [12].

THEOREM 2.5.

$$\lambda_{d+1}^{-1/2}(2m - j) \leqq \Lambda(E, 2m, z, j) \leqq \gamma^2 K_{m,2m,z,j}(\bar{\Delta})^{2m-j}, \tag{2.27}$$

where

$$d \equiv \dim \{D^j[S(E, \Delta, z)]\} \tag{2.28}$$

and

$$K_{m,2m,z,j} \equiv (K_{m,m,z,j})(K_{m,m,z,0}), \tag{2.29}$$

for all γ-elliptic operators E, $0 \leqq M$, $\Delta \in \mathcal{P}_M$, $m - 1 \leqq z \leqq 2m - 2$, and $0 \leqq j \leqq m$.

Proof. Applying the Cauchy-Schwarz inequality to the second integral relation and using the γ-ellipticity of E, we have

$$\|D^m(f - \mathcal{I}f)\|_{0,2}^2 \leqq \gamma^2 e(f - \mathcal{I}f, f - \mathcal{I}f) \leqq \gamma^2 \|E(f)\|_{0,2} \|f - \mathcal{I}f\|_{0,2} . \tag{2.30}$$

Applying the proof of Theorem 2.4, we have

$$\|D^j(f - \mathcal{I}f)\|_{0,2} \leqq K_{m,m,z,j} \|D^m(f - \mathcal{I}f)\|_{0,2} (\bar{\Delta})^{m-j} . \tag{2.31}$$

Using (2.31) for the special case of $j = 0$ in (2.30) yields (2.32)

$$\|D^m(f - \mathcal{I}f)\|_{0,2} \leqq \gamma^2 \|E(f)\|_{0,2} K_{m,m,z,0} (\bar{\Delta})^m . \tag{2.32}$$

Using (2.32) to bound the right-hand side of (2.31) gives us the right-hand inequality of (2.27). The left-hand inequality of (2.27) follows as in Theorem 2.4. Q.E.D.

Now we obtain *explicit upper* bounds for the quantities $\Lambda_\infty(E, p, z, j)$, $1 \leqq m$, $p = m$ or $2m$, $m - 1 \leqq z \leqq 2m - 2$, and $0 \leqq j \leqq m$, defined by

$$\Lambda_\infty(E, m, z, j) \equiv \sup \{\|D^j(f - \mathcal{S}f)\|_{0,\infty}/[e(f, f)]^{1/2} \mid f \in W^{m,2}, e(f, f) \neq 0\} \tag{2.33}$$

and

$$\Lambda_\infty(E, 2m, z, j) \equiv \sup \{\|D^j(f - \mathcal{S}f)\|_{0,\infty}/\|E(f)\|_{0,2} \mid f \in W^{2m,2}, \|E(f)\|_{0,2} \neq 0\} . \tag{2.34}$$

As a generalization of Theorem 5.1 of [11] and Theorem 6 of [12], we have

THEOREM 2.6.

$$\Lambda_\infty(E, m, z, j) \leqq \gamma K^\infty_{m,m,z,j}(\bar{\Delta})^{m-j-1/2} , \tag{2.35}$$

where

$$K^\infty_{m,m,z,j} \equiv \begin{cases} K_{m,m,z,j+1} , & \text{if } m - 1 = z , \quad 0 \leqq j \leqq m - 1 , \\ K_{m,m,z,j+1} , & \text{if } m - 1 < z \leqq 2m - 2 , \\ & \quad 0 \leqq j - 2m - 2 - z , \\ (j - 2m + 3 + z)^{1/2} K_{m,m,z,j+1} , & \text{if } m - 1 < z \leqq 2m - 2 , \\ & \quad 2m - 2 - z < j \leqq m - 1 , \end{cases} \tag{2.36}$$

for all γ-elliptic operators E, $0 \leqq M$, $\Delta \in \mathcal{P}_M$, $m - 1 \leqq z \leqq 2m - 2$, and $0 \leqq j \leqq m - 1$.

Proof. We give the proof in the special case of $m - 1 = z$, $0 \leqq j \leqq m - 1$, as the proof in the other cases is analogous. Given any $x \in [a, b]$, there exists a point $y \in [a, b]$ such that $D^j(f - \mathcal{S}f)(y) = 0$ and $|x - y| \leqq \bar{\Delta}$. Hence,

$$D^j(f - \mathcal{S}f)(x) = \int_y^x D^{j+1}(f - \mathcal{S}f)(t)dt$$

and

$$\|D^j(f - \mathcal{S}f)\|_{0,\infty} \leqq (\bar{\Delta})^{1/2}\|D^{j+1}(f - \mathcal{S}f)\|_{0,2} .$$

The result now follows from applying Theorem 2.4 to the right-hand side of the preceding inequality. Q.E.D.

As in Theorem 2.6, we have the following result, as a generalization of Theorem 5.2 of [11] and Theorem 8 of [12].

THEOREM 2.7.

$$\Lambda_\infty(E, 2m, z, j) \leqq \gamma^2 K^\infty_{m,2m,z,j}(\bar{\Delta})^{2m-j-1/2} , \tag{2.37}$$

where

$$K^\infty_{m,2m,z,j+1} \equiv \begin{cases} K_{m,2m,z,j+1} , & \text{if } m - 1 = z , \quad 0 < j \leqq m - 1 , \\ K_{m,2m,z,j+1} , & \text{if } m - 1 < z \leqq 2m - 2 , \\ & \quad 0 \leqq j \leqq 2m - 2 - z , \\ (j - 2m + 3 + z)^{1/2} K_{m,2m,z,j+1} , & \text{if } m - 1 < z \leqq 2m - 2 , \\ & \quad 2m - 2 - z < j, \end{cases} \tag{2.38}$$

for all γ-*elliptic operators* E, $0 \leqq M$, $\Delta \in \mathcal{P}_M$, $m - 1 \leqq z \leqq 2m - 2$, *and* $0 \leqq j \leqq m - 1$.

In some special cases the preceding results may be strengthened. Let

$$\Lambda(D^{2m}, p, z, j), \quad 1 \leqq m, m \leqq p \leqq 2m, m - 1 \leqq z \leqq 2m - 2$$

be defined by

$$\Lambda(D^{2m}, p, z, j) \equiv \sup \{\|D^j(f - \mathcal{I}f)\|_{0,2}/\|D^p f\|_{0,2} | f \in W^{p,2}, \|D^p f\|_{0,2} \neq 0\} . \tag{2.39}$$

THEOREM 2.8.

$$\lambda_{d+1}^{-1/2}(p - j) \leqq \Lambda(D^{2m}, p, z, j) \leqq K_{m,p,z,j}(\bar{\Delta})^{p-j} . \tag{2.40}$$

where

$$d \equiv \dim \{D^j[S(D^{2m}, \Delta, z)]\} \tag{2.41}$$

and

$$K_{m,p,z,j} \equiv \left\{K_{p,p,2m-1,z} + K_{m,2m,z,j} \cdot 2^{(1/2)(2m-p)}\left[\frac{p!}{(2p - 2m)!}\right]^2 \left(\frac{\bar{\Delta}}{\underline{\Delta}}\right)^{2m-p}\right\} \tag{2.42}$$

for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathcal{P}_M$, $m < p < 2m$, $4m - 2p - 1 \leqq z \leqq 2m - 2$, *and* $0 \leqq j \leqq m$.

Proof. See Theorem 3.6 of [11]. Q.E.D.

Let $\Lambda_\infty(D^{2m}, p, z, j)$, $1 \leqq m, m \leqq p \leqq 2m, m - 1 \leqq z \leqq 2m - 2$ be defined by

$$\Lambda_\infty(D^{2m}, p, z, j) \equiv \sup \{\|D^j(f - \mathcal{I}f)\|_{0,\infty}/\|D^p f\|_{0,2} | f \in W^{p,2} \|D^p f\|_{0,2} \neq 0\} . \tag{2.43}$$

THEOREM 2.9.

$$\Lambda_\infty(D^{2m}, p, z, j) \leqq K^\infty_{m,p,z,j}(\bar{\Delta})^{p-j-1/2} , \tag{2.44}$$

where

$$K^\infty_{m,p,z,j} \equiv \left\{K^\infty_{p,p,2m-1,j} + K^\infty_{m,2m,z,j} \cdot 2^{(1/2)(2m-p)}\left[\frac{p!}{(2p - 2m)!}\right]^2 \left(\frac{\bar{\Delta}}{\underline{\Delta}}\right)^{2m-p}\right\} , \tag{2.45}$$

for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathcal{P}_M$, $m < p < 2m$, $4m - 2p - 1 \leqq z \leqq 2m - 2$, *and* $0 \leqq j \leqq m - 1$.

Proof. See Theorem 5.3 of [11]. Q.E.D.

Let $\Lambda_{\infty,\infty}(D^{2m}, p, z, j)$, $1 \leqq m, m \leqq p \leqq 2m, m - 1 \leqq z \leqq 2m - 2$ be defined by

$$\begin{aligned} &\Lambda_{\infty,\infty}(D^{2m}, p, z, j) \\ &\qquad \equiv \sup \{\|D^j(f - \mathcal{I}f)\|_{0,\infty}/\|D^p f\|_{0,\infty} | f \in W^{p,\infty}, \quad \|D^p f\|_{0,\infty} \neq 0\} . \end{aligned} \tag{2.46}$$

THEOREM 2.10. *There exists a positive constant, K, such that*

$$\Lambda_{\infty,\infty}(D^{2m}, p, m - 1) \leqq K(\bar{\Delta})^{p-j} , \tag{2.47}$$

for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathcal{P}_M$, $m \leqq p \leqq 2m$, *and* $0 \leqq j \leqq m - 1$.

Using a result of M. E. Rose, we can prove an analogue of Theorem 2.10 for second-order γ-elliptic operators. If E is a second-order γ-elliptic operator, let

$$\Lambda_{\infty,\infty}(E, 2, 0, 0) \equiv \sup \{\|f - \mathcal{I}f\|_{0,\infty}/\|D^2 f\|_{0,\infty} | f \in W^{2,\infty}, \|D^2 f\|_{0,\infty} \neq 0\} . \tag{2.48}$$

THEOREM 2.11. *If E is a second-order γ-elliptic operator, there exists a positive constant, K, such that*

$$\Lambda_{\infty,\infty}(E, 2, 0, 0) \leqq K(\bar{\Delta})^2, \tag{2.49}$$

for all $0 \leqq M$ *and* $\Delta \in \mathcal{P}_M$.

Proof. By a theorem of M. E. Rose, cf. [10, p. 183],

$$(f - \mathcal{I}f)(x) = \int_a^b H_\Delta(x, y)E(f)(y)dy, \quad \text{for all } \Delta \in \mathcal{P}_M(a, b),\ f \in K^{2,\infty}[a, b],$$

where $H_\Delta(x, y) \equiv \sum_{i=0}^M K_i(x, y)$ and $K_i(x, y)$ equals the Green's function for $E[u](x)$ on $[x_i, x_{i+1}]$ subject to the boundary conditions $u(x_i) = u(x_{i+1}) = 0$ if x, $y \in [x_i, x_{i+1}]$ and $K_i(x, y) = 0$ if either or both $x, y \notin [x_i, x_{i+1}]$. Thus, if

$$x \in [x_i, x_{i+1}],\ |(f - \mathcal{I}f)(x)| = \left|\int_{x_i}^{x_{i+1}} K_i(x, y)E(f)(y)dy\right|$$
$$\leqq \|K_i(x, y)\|_{L^\infty[x_i, x_{i+1}]} \|E(f)\|_{0,\infty}(\bar{\Delta}).$$

Hence, it suffices to find a bound for $K_i(x, y)$, $0 \leqq i \leqq M$, given that x and $y \in [x_i, x_{i+1}]$.

Following [6], we have that

$$K_i(x, y) = -v_1(x)v_2(y)/C \quad \text{for } x \leqq y,$$

where $E[v_1](x) = E[v_2](x) = 0$, for all $x \in [x_i, x_{i+1}]$, $v_1(x_i) = 0$, $Dv_1(x_i) = 1$, $v_2(x_i) = 1$, $v_2(x_{i+1}) = 0$, and

$$C \equiv \begin{vmatrix} v_1(x) & v_2(x) \\ Dv_1(x) & Dv_2(x) \end{vmatrix}$$

is a constant, and $K_i(x, y) = K_i(y, x)$ for all $x \geqq y$. If $w_2 \equiv p(x)Dv_1(x)$, then

$$D\begin{bmatrix} v_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 & -\dfrac{1}{p(x)} \\ q(x) & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1(x_i) \\ w_2(x_i) \end{bmatrix} = \begin{bmatrix} 0 \\ p(x_i) \end{bmatrix}$$

and if $u_2(x) \equiv w_2(x) - p(x_i)$,

$$D\begin{bmatrix} v_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & -\dfrac{1}{p(x)} \\ q(x) & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} p(x_i)/p(x) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} v_1(x_i) \\ u_2(x_i) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It is easy to verify that

$$\begin{bmatrix} v_1(x) \\ u_2(x) \end{bmatrix} = \int_{x_i}^x \exp\left(-\int_s^x A(t)dt\right)\begin{bmatrix} -p(x_i)/p(x) \\ 0 \end{bmatrix} ds,$$

where

$$A(t) \equiv \begin{bmatrix} 0 & -\dfrac{1}{p(t)} \\ q(t) & 0 \end{bmatrix},$$

and hence

$$\begin{aligned} |v_1(x)| &\leqq [v_1^2(x) + u_2^2(x)]^{1/2} \\ &\leqq \max_{x,s \in [a,b]} \left\{ \left\| \exp\left[-\int_S^x A(t)dt \right] \right\|_2 \cdot \left\| \begin{matrix} -p(x_i)/p(x) \\ 0 \end{matrix} \right\|_2 \right\} \cdot (x_{i+1} - x_i), \end{aligned}$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Thus, we have shown that there exists a positive constant, K, such that $|v_1(x)| \leqq K(\bar{\Delta})$ Similarly, we may show that there exists a positive constant, K, such that $|v_2(y)| \leqq K$ and hence $|K_i(x, y)| \leqq (K^2/c)(\bar{\Delta})$. Q.E.D.

3. Nonlinear Two-Point Boundary Value Problems. We consider the differential equation

$$E[u(x)] = f(x, u(x)), \quad a \leqq x \leqq b, \tag{3.1}$$

subject to the boundary conditions

$$D^k u(a) = D^k u(b) = 0, \quad 0 \leqq k \leqq m-1, \tag{3.2}$$

where

$$E[u(x)] \equiv \sum_{j=0}^{m} (-1)^j D^j [p_j(x) D^j u(x)], \quad p_j(x) \in W^{j,2} \cap W^{0,\infty}, \quad 0 \leqq j \leqq m,$$

$p_m(x) \geqq w > 0$ for all $x \in [a, b]$, and there exists a positive constant, γ, such that

$$\|D^m w\|^2_{L^2[a,b]} \leqq \gamma^2 \int_a^b \sum_{j=0}^{m} p_j(x)[D^j w(x)]^2 dx \tag{3.3}$$

for all $w \in W_0^{m,2}$.

Let

$$\Lambda \equiv \inf_{w \in W_0^{m,2};\, w \neq 0} \frac{\int_a^b \sum_{j=0}^{n} p_j(x)[D^j w(x)]^2 dx}{\int_a^b [w(x)]^2 dx} > 0 \tag{3.4}$$

and $f(x, u)$ be a real-valued function defined on $[a, b] \times R$, measurable in x for every u in R and such that

$$\frac{f(x, u) - f(x, v)}{u - v} \leqq \mu < \Lambda \tag{3.5}$$

for almost all $x \in [a, b]$ and all $u, v \in R$ with $u \neq v$ and for each $c > 0$ there exists a finite constant $M(c)$ such that

$$\left| \frac{f(x, u) - f(x, v)}{u - v} \right| \leqq M(c) \tag{3.6}$$

for almost all $x \in [a, b]$ and all $u, v \in R$ with $|u| \leqq c$, $|v| \leqq c$, and $u \neq v$.

As in [3], [4], [7], and [9], we consider the Rayleigh-Ritz-Galerkin procedure for approximating the solution to (3.1)—(3.2), i.e., if S is an N-dimensional subspace of $W_0^{m,2}$ with linearly independent basis functions $\{B_i(x)\}_{i=1}^N$, we seek an approximation of the form $u_S \equiv \sum_{i=1}^N \beta_i B_i(x)$, where the coefficients $\{\beta_i\}_{i=1}^N$ are determined by the nonlinear, algebraic system

$$A\boldsymbol{\beta} = \mathbf{F}(\boldsymbol{\beta}) , \tag{3.7}$$

where

$$A \equiv \left(\int_a^b \sum_{k=0}^m p_k(x) D^k B_i(x) D^k B_j(x) dx \right)$$

and

$$\mathbf{F}(\boldsymbol{\beta}) \equiv \left[\int_a^b f\left(x, \sum_{i=1}^N \beta_i B_i(x)\right) B_j(x) dx \right] .$$

From [3] and [4], we have the following fundamental result.

THEOREM 3.1. *With the preceding hypotheses, there exists a unique generalized solution, u, to* (3.1)—(3.2) *over $W_0^{m,2}$,*

$$e^{1/2}(u, u) \leqq \gamma \frac{\Lambda^{1/2}}{\Lambda - \mu} \|f(x, 0)\|_{0,2} , \tag{3.8}$$

the Rayleigh-Ritz-Galerkin procedure gives a unique approximation, u_S, and

$$e^{1/2}(u_S, u_S) \leqq \gamma \frac{\Lambda^{1/2}}{\Lambda - u} \|f(x, 0)\|_{0,2} . \tag{3.9}$$

The purpose of this section is to discuss the size of the error for the special cases of elliptic spline subspaces which are somehow related to the differential operator E. We begin with the linear case, i.e., $f(x, u)$ is independent of u, and subspaces of E-splines. Generalizing [7], [9], and [10], we have

THEOREM 3.2. *If $f(x, u)$ is independent of u, $\Delta \in \mathcal{P}$, $m - 1 \leqq z \leqq 2m - 2$, and $S \equiv S(E, \Delta, z) \cap W_0^{m,2}$, then $u_S = \mathcal{I}u$, where $\mathcal{I}$ is the interpolation mapping of $C^{m-1}[a, b]$ into $S(E, \Delta, z)$.*

Proof. Using the notation of Section 1, it is easy to verify that $e(u_S, B_j) = (f, B_j)_{L^2(a,b)}$ and $e(u, B_j) = (f, B_j)_{L^2(a,b)}$ for all $1 \leqq j \leqq N$. Hence,

$$e(u_S - u, B_j) = 0 \quad \text{for all } 1 \leqq j \leqq N . \tag{3.10}$$

Moreover, from the Corollary to Theorem 2.1,

$$e(u - \mathcal{I}u, B_j) = 0 \quad \text{for all } 1 \leqq j \leqq N \tag{3.11}$$

and hence

$$e(u_S - \mathcal{I}u, B_j) = 0 \quad \text{for all } 1 \leqq j \leqq N$$

and

$$0 = e(u_S - \mathcal{I}u, u_S - \mathcal{I}) \geqq \gamma \|u_S - \mathcal{I}u\|_{m,2}^2 . \qquad \text{Q.E.D.}$$

Combining Theorem 3.2 with the results of Section 2 and Theorem 2 of [2], we obtain the following

COROLLARY. *Let $f(x, u)$ be independent of u.*

(i) *If $u \in W^{2m,2}$ and $S \equiv S(E, \Delta, z) \cap W_0^{m,2}$, then*

$$\|u - u_S\|_{j,2} \leqq \gamma^2 K_{m,2m,z,j} \|f(x)\|_{0,2} (\bar{\Delta})^{2m-j}, \qquad 0 \leqq j \leqq m,$$

and

$$\|u - u_S\|_{j,\infty} \leqq \gamma^2 K^{\infty}_{m,2m,z,j} \|f(x)\|_{0,2} (\bar{\Delta})^{2m-j-1/2}, \qquad 0 \leqq j \leqq m - 1,$$

for all $\Delta \in \mathcal{P}$ and $m - 1 \leqq z \leqq 2m - 2$.

(ii) *If $u \in W^{p,2}$, $E \equiv D^{2m}$, and $S \equiv S(D^{2m}, \Delta, z) \cap W_0^{m,2}$, then*

$$\|u - u_S\|_{j,2} \leqq K_{m,p,z,j} \|D^p u\|_{0,2} (\bar{\Delta})^{p-j}, \qquad 0 \leqq j \leqq m,$$

and

$$\|u - u_S\|_{j,\infty} \leqq K^{\infty}_{m,p,z,j} \|D^p u\|_{0,2} (\bar{\Delta})^{p-j-1/2}, \qquad 0 \leqq j \leqq m - 1,$$

for all $\Delta \in \mathcal{P}$, $m < p < 2m$, and $4m - 2p - 1 \leqq z \leqq 2m - 2$.

(iii) *Let $\mathcal{P} \equiv D^{2m}$ and $S \equiv S(D^{2m}, \Delta, m - 1) \cap W_0^{m,2}$. If $u \in W^{p,2}$,*

$$\|u - u_S\|_{j,2} \leqq c_{j,m,p,2,2} \|D^p u\|_{0,2} (\bar{\Delta})^{p-j}, \qquad 0 \leqq j \leqq m,$$

for all $\Delta \in \mathcal{P}$ and $m \leqq p \leqq 2m$, where $c_{j,m,p,2,2}$ is the constant defined in Eq. (2.10) *of* [2]. *If $u \in W^{p,\infty}$,*

$$\|u - u_S\|_{j,\infty} \leqq c_{j,m,p,2,2} \|D^p u\|_{0,\infty} (\bar{\Delta})^{p-j} \qquad 0 \leqq j \leqq m,$$

for all $\Delta \in \mathcal{P}$ and $m \leqq p \leqq 2m$.

(iv) *If $m = 1$, $u \in W^{2,\infty}$, and $S \equiv S(E, \Delta, 0) \cap W_0^{1,2}$, there exists a positive constant, K, such that $\|u - u_S\|_{0,\infty} \leqq K(\bar{\Delta})^2$, for all $\Delta \in \mathcal{P}$.*

Now we consider the nonlinear case of Eq. (3.1). The following theorem is a generalization of analogous results of [7] and [9]. We remark that its proof is considerably simpler than the proofs of the referenced results.

THEOREM 3.3. *If* (3.1)—(3.7) *hold and $S \equiv S(E, \Delta, z) \cap W_0^{m,2}$, then*

$$e^{1/2}(\mathcal{G}u - u_S, \mathcal{G}u - u_S) \leqq \frac{\Lambda^{1/2}}{\Lambda - \mu} \|f(x, u) - f(x, \mathcal{G}u)\|_{0,2}, \tag{3.12}$$

for all $\Delta \in \mathcal{P}$ and $m - 1 \leqq z \leqq 2m - 2$.

Proof. Using the above hypotheses and the Corollary to Theorem 2.1, we have

$$\begin{aligned}
\left(1 - \frac{\mu}{\Lambda}\right) e(\mathcal{G}u - u_S, \mathcal{G}u - u_S) &\leqq e(\mathcal{G}u - u_S, \mathcal{G}u - u_S) - \mu \int_a^b (\mathcal{G}u - u_S)^2 dx \\
&\leqq e(\mathcal{G}u - u_S, \mathcal{G}u - u_S) \\
&\quad - \int_a^b \left[\frac{f(x, \mathcal{G}u) - f(x, u_S)}{\mathcal{G}u - u_S}\right] (\mathcal{G}u - u_S)^2 dx \\
&= e(\mathcal{G}u, \mathcal{G}u - u_S) - \int_a^b f(x, \mathcal{G}u)(\mathcal{G}u - u_S) dx \\
&= e(u, \mathcal{G}u - u_S) - \int_a^b f(x, \mathcal{G}u)(\mathcal{G}u - u_S) dx \\
&= \int_a^b [f(x, u) - f(x, \mathcal{G}u)](\mathcal{G}u - u_S) dx .
\end{aligned}$$

The result follows by applying the Cauchy-Schwarz inequality and definition (3.4). Q.E.D.

COROLLARY. *If* (3.1)—(3.7) *hold and* $S \equiv S(E, \Delta, z) \cap W_0^{m,2}$, *then*

$$\|D^m(\mathcal{J}u - u_S)\|_{L^2(a,b)} \leqq \frac{\gamma\Lambda^{1/2}}{\Lambda - \mu} M(c)\|u - \mathcal{J}u\|_{0,2} \quad \text{for all } \Delta \in \mathcal{P} \tag{3.2}$$

and $m - 1 \leqq z \leqq 2m - 2$, *where,*

$$c \equiv \frac{1}{2} \frac{(b-a)^{m-1/2}}{\pi^{m-1}} \gamma \frac{\Lambda^{1/2}}{\Lambda - \mu} \|f(x, 0)\|_{0,2} \tag{3.13}$$

and $M(c)$ *is given by* (3.6).

Proof. By (2.5), $e(\mathcal{J}u, \mathcal{J}u) \leqq e(u, u)$. Hence, using (3.8), we have

$$e(\mathcal{J}u, \mathcal{J}u) \leqq e(u, u) \leqq \frac{\Lambda^{1/2}}{\Lambda - \mu} \|f(x, 0)\|_{0,2}$$

and thus by the Rayleigh-Ritz inequality, cf. [5, p. 184], and the inequality $\|w\|_{0,\infty} \leqq \frac{1}{2}(b - a)^{1/2}\|Dw\|_{0,2}$, for all $w \in W_0^{m,2}$, $\|\mathcal{J}u\|_{0,\infty} \leqq c$ and $\|u\|_{0,\infty} \leqq c$, where c is given in (3.13). The result now follows by combining these a priori bounds with (3.3), (3.6), and (3.12). Q.E.D.

From the Rayleigh-Ritz and triangle inequalities and the preceding result, we have

THEOREM 3.4. *If* (3.1)—(3.7) *hold and* $S \equiv S(E, \Delta, z) \cap W_0^{m,2}$, *then*

$$\begin{aligned}\|D^j(u - u_S)\|_{0,2} \leqq \|D^j(u - \mathcal{J}u)\|_{0,2} &+ \left(\frac{b-a}{\pi}\right)^{m-j} \frac{\gamma\Lambda^{1/2}}{\Lambda - \mu} M(c) \\ &\cdot \|u - \mathcal{J}u\|_{0,2}, \qquad 0 \leqq j \leqq m,\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\|D^j(u - u_S)\|_{0,\infty} \leqq \|D^j(u - \mathcal{J}u)\|_{0,\infty} &+ \frac{1}{2} \frac{(b-a)^{m-j-1/2}}{\pi^{m-j}} \frac{\gamma\Lambda^{1/2}}{\Lambda - \mu} M(c) \\ &\cdot \|u - \mathcal{J}u\|_{0,\infty}, \qquad 0 \leqq j \leqq m - 1,\end{aligned} \tag{3.15}$$

for all $\Delta \in \mathcal{P}$ *and* $m - 1 \leqq z \leqq 2m - 2$, *where the constant* c *is given by* (3.13) *and* $M(c)$ *by* (3.6).

We remark that as was done in the Corollary to Theorem 3.2, we can now combine the results of Theorem 3.3 with those of Section 1 to obtain specific error bounds. We leave the details to the reader.

As in [3], [4], [7], and [9], one expects that more accurate Rayleigh-Ritz-Galerkin approximations are possible if the solution u is particularly smooth, i.e., $D^j u$ exists for some $j > 2m$ and if we use a "spline-type" space of "higher degree" than $S(E, \Delta, z)$. We now generalize and simplify the construction and proofs of [7] and [9].

For each positive integer, t, we consider the differential operator

$$E^t[u] \equiv \sum_{j=0}^{m} D^{j+2t}[p_j(x)D^{j+2t}u(x)]. \tag{3.16}$$

From inequality (3.3), we have that

$$\|D^{m+2t}w\|_{0,2} \leqq \gamma^2 \int_a^b \sum_{j=0}^{m} p_j(x)[D^{j+2t}w(x)]^2 dx \equiv \gamma^2 e_t(w, w) ,$$

for all $w \in W_0^{m+2t,2}$. Thus, following Section 1, we may define the space $S(E^t, \Delta, z) \subset W^{m+2t,2}$, for all $\Delta \in \mathcal{P}$ and $m + 2t - 1 \leqq z \leqq 2m + 2t - 2$.

THEOREM 3.5. *If* (3.1)—(3.7) *hold and* $S \equiv D^{2t}S(E^t, \Delta, z) \cap W_0^{m,2}$, *then there exists a positive constant,* K, *such that*

$$\|D^j(u - u_S)\|_{0,2} \leqq K\|D^{j+2t}(\mathcal{I}y - y)\|_{0,2} , \qquad 0 \leqq j \leqq m , \tag{3.17}$$

and

$$\|D^j(u - u_S)\|_{0,\infty} \leqq K\|D^{j+2t}(\mathcal{I}y - y)\|_{0,\infty} , \qquad 0 \leqq j \leqq m - 1 , \tag{3.18}$$

where

$$y(x) \equiv \int_a^x \int_a^{x_{2t}} \cdots \int_a^{x_2} u(x_1)dx_1dx_2 \cdots dx_{2t} \equiv \mathcal{M}(u)$$

and $\mathcal{I}$ *is the interpolation mapping into* $S(E^t, \Delta, z)$, *for all* $\Delta \in \mathcal{P}$, $m + 2t - 1 \leqq z \leqq 2m + 2t - 2$.

Proof. We prove only (3.17) as the proof of (3.18) is essentially identical. As in Theorem 3.4, it suffices to show that there exists a positive constant, K, such that

$$\|D^m(u_s - D^{2t}\mathcal{I}y)\|_{0,2} \leqq K\|D^{2t}(y - \mathcal{I}y)\|_{0,2} .$$

As in the proof of Theorem 3.3,

$$\begin{aligned}
&\left(1 - \frac{\mu}{\Lambda}\right)e(D^{2t}\mathcal{I}y - u_S, D^{2t}\mathcal{I}y - u_S) \\
&\leqq e(D^{2t}\mathcal{I}y - u_S, D^{2t}\mathcal{I}y - u_S) \quad - \mu \int_a^b (D^{2t}\mathcal{I}y - u_S)^2 dx \\
&\leqq e(D^{2t}\mathcal{I}y, D^{2t}\mathcal{I}y - u_S) \quad - \int_a^b f(x, D^{2t}\mathcal{I}y)(D^{2t}\mathcal{I}y - u_S)dx \\
&= e_t(\mathcal{I}y, \mathcal{I}y - \mathcal{M}(u_S)) \quad - \int_a^b f(x, D^{2t}\mathcal{I}y)(D^{2t}\mathcal{I}y - u_S)dx \\
&= e_t(y, \mathcal{I}y - \mathcal{M}(u_S)) \quad - \int_a^b f(x, D^{2t}\mathcal{I}y)(D^{2t}\mathcal{I}y - u_S)dx \\
&= \int_a^b [f(x, D^{2t}y) - f(x, D^{2t}\mathcal{I}y)]\,(D^{2t}\mathcal{I}y - u_S)dx .
\end{aligned}$$

Applying the triangle inequality and (3.4), we have

$$e^{1/2}(D^{2t}\mathcal{I}y - u_S, D^{2t}\mathcal{I}y - u_S) \leqq \frac{\Lambda^{1/2}}{\Lambda - \mu}\,\|f(x, D^{2t}y) - f(x, D^{2t}\mathcal{I}y)\|_{0,2} .$$

The required result follows in the same way as the Corollary to Theorem 3.3, since

$$\begin{aligned}\|D^{2t}\mathcal{I}y\|_{0,\infty} &\leqq \|D^{2t}y - D^{2t}\mathcal{I}y\|_{0,\infty} + \|D^{2t}y\|_{0,\infty} \\ &\leqq \frac{1}{2}\frac{(b-a)^{m-1/2}}{\pi^{m-1}}\|D^{m+2t}(y-\mathcal{I}y)\|_{0,2} + \|D^{2t}y\|_{0,\infty} \\ &\leqq \frac{(b-a)^{m-1/2}\gamma}{2\pi^{m-1}} e_t^{1/2}(y-\mathcal{I}y, y-\mathcal{I}y) + \|D^{2t}y\|_{0,\infty} \\ &\leqq \frac{(b-a)^{m-1/2}\gamma}{2\pi^{m-1}} e_t^{1/2}(y,y) + \|D^{2t}y\|_{0,\infty}\,,\end{aligned}$$

by Theorem 2.2. Q.E.D.

We now state an analogue of part (i) of the Corollary of Theorem 3.2 to illustrate the power of the result of Theorem 3.5. The details of the analogues of the other parts of the Corollary of Theorem 3.2 are similar and are left to the reader.

COROLLARY. *Let* (3.1)—(3.7) *hold and* $S \equiv D^{2t}S(E^t, \Delta, z) \cap W_0^{m,2}$. *If* $u \in W^{2m+2t,2}$ *then there exists a positive constant,* K, *such that*

$$\|u - u_S\|_{j,2} \leqq K(\bar{\Delta})^{2m+2t-j}\|u\|_{2m+2t,2}\,, \qquad 0 \leqq j \leqq m\,,$$

and

$$\|u - u_S\|_{j,\infty} \leqq K(\bar{\Delta})^{2m+2t-j+1/2}\|u\|_{2m+2t,2}\,, \qquad 0 \leqq j \leqq m\,.$$

Proof. Since $y \equiv \mathfrak{M}(u) \in W^{2m+4t,2}$, the results follow from Theorems 2.5, 2.7, and 3.5. Q.E.D.

Finally, we make some remarks about the use of subspaces formed from low-order perturbations of the differential operator E. If we wish to solve (3.1)—(3.2) and $\tilde{E}[u] \equiv \sum_{j=0}^{m} D^j[q_j(x)D^ju]$ is such that $q_j(x) = p_j(x)$, $1 \leqq j \leqq m$, and $q_0(x)$ is such that there exists a $\eta > 0$ such that

$$\|D^m w\|_{0,2} \leqq \eta^2 \int_a^b \sum_{j=0}^{m} q_j(x)[D^j w(x)]^2 dx \quad \text{for all } w \in W_0^{m,2}\,,$$

then we rewrite (3.1) as

$$\tilde{E}[u] = f(x,u) + \tilde{E}[u] - E[u] \equiv g(x,u)\,. \tag{3.19}$$

It is easy to verify that if S is a finite-dimensional subspace of $W_0^{m,2}$, then the Rayleigh-Ritz-Galerkin equations for S for the problem (3.19)—(3.2) are identical to those for the problem (3.1)—(3.2). Hence, if we use a subspace of the form $S \equiv D^{2t}S(\tilde{E}^t, \Delta, z) \cap W_0^{m,2}$ for some $t \geqq 0$, $\Delta \in \mathcal{P}$, and $m - 1 \leqq z \leqq 2m - 2$, the preceding analysis applies. The details are left to the reader.

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