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On Amicable and Sociable Numbers*

By Henri Cohen

Abstract. An exhaustive search has yielded 236 amicable pairs of which the lesser number is smaller than 10^8, 57 pairs being new.

It has also yielded 9 new sociable groups of order 10 or less, of which the lesser number is smaller than 6.10^7; the 9 sociable groups are all of order 4.

The sequence of iterates of the function $s(n) = \sigma(n) - n$ starting with 276 has also been extended to 119 terms.

Introduction. Let $n \geqq 2$ be an integer, and

$$\sigma(n) = \sum_{d|n} d, \qquad s(n) = \sigma(n) - n = \sum_{d|n;d \neq n} d.$$

We wish to study the behavior of the sequence:

$$a_0(n) = n$$
$$a_{k+1}(n) = s(a_k(n))$$

which will be called the aliquot series of n.

It is clear that if this sequence is bounded as $k \to \infty$ it is periodic, since $a_k(n)$ can take only a finite number of values.

Consequently the sequence can have essentially three distinct behaviors:

(a) The sequence converges, i.e. there exists a k for which $a_k = 1$ (or equivalently a_{k-1} prime).

(b) The sequence is periodic of period t: there exists k_0 such that

$$a_{k+t} = a_k \quad \text{for all } k \geqq k_0.$$

If one can take $k_0 = 0$, the sequence is purely periodic; in this case: if $t = 1$, n is a perfect number, if $t = 2$, $(n, s(n))$ is a pair of amicable numbers, and in general the t-uplet $(a_0, \ldots, a_{t-1})$ is a sociable group of order t.

(c) The sequence is unbounded.

Results on Amicable Numbers. Two recent papers [6], [7], listed all pairs of amicable numbers up to 10^6 and 10^7 respectively.

Table 1 extends these lists and contains all amicable pairs with the lesser number between 10^7 and 10^8. The 57 pairs marked with an asterisk are not found in the lists given by Escott [1], Poulet [2], Garcia [3], Lee [4], Lee [5], and seem to be new.

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TABLE 1

AMICABLE NUMBERS

	$10254970 = 2.5.11.53.1759$	$10273670 = 2.5.11.59.1583$
	$10533296 = 2^4.19.34649$	$10949704 = 2^3.29.109.433$
	$10572550 = 2.5^2.19.31.359$	$10854650 = 2.5^2.31.47.149$
	$10596368 = 2^4.29.41.557$	$11199112 = 2^3.53.61.433$
*	$10634085 = 3^4.5.7.11^2.31$	$14084763 = 3.7.11^2.23.241$
*	$10992735 = 3^2.5.13.19.23.43$	$12070305 = 3^2.5.13.47.439$
*	$11173460 = 2^2.5.53.83.127$	$13212076 = 2^2.31.47.2267$
*	$11252648 = 2^3.11.71.1801$	$12101272 = 2^3.67.107.211$
	$11498355 = 3^4.5.11.29.89$	$12024045 = 3^4.5.11.2699$
	$11545616 = 2^4.19.163.233$	$12247504 = 2^4.491.1559$
	$11693290 = 2.5.7.167047$	$12361622 = 2.7^2.13.31.313$
	$11905504 = 2^5.13.28619$	$13337336 = 2^3.107.15581$
	$12397552 = 2^4.23.59.571$	$13136528 = 2^4.359.2287$
*	$12707704 = 2^3.17.41.43.53$	$14236136 = 2^3.107.16631$
*	$13671735 = 3.5.7^2.11.19.89$	$15877065 = 3.5.17.19.29.113$
*	$13813150 = 2.5^2.13.79.269$	$14310050 = 2.5^2.29.71.139$
	$13921528 = 2^3.19.67.1367$	$13985672 = 2^3.19.101.911$
	$14311688 = 2^3.17.47.2239$	$14718712 = 2^3.23.167.479$
*	$14426230 = 2.5.7.13.83.191$	$18087818 = 2.7.31.71.587$
	$14443730 = 2.5.7^3.4211$	$15882670 = 2.5.19.179.467$
	$14654150 = 2.5^2.7.149.281$	$16817050 = 2.5^2.179.1879$
	$15002464 = 2^5.37.12671$	$15334304 = 2^5.227.2111$
*	$15363832 = 2^3.11.71.2459$	$16517768 = 2^3.53.163.239$
	$15938055 = 3^2.5.7.19.2663$	$17308665 = 3^2.5.11.73.479$
*	$16137628 = 2^2.13.23.103.131$	$16150628 = 2^2.13.31.43.233$
	$16871582 = 2.7^2.13.17.19.41$	$19325698 = 2.7^2.19.97.107$
*	$17041010 = 2.5.7.31.7853$	$19150222 = 2.7.13.43.2447$
*	$17257695 = 3.5.7.13.47.269$	$17578785 = 3.5.7.23.29.251$
	$17754165 = 3^2.5.11.13.31.89$	$19985355 = 3^2.5.13.127.269$
	$17844255 = 3^2.5.11.13.47.59$	$19895265 = 3^2.5.13.71.479$
	$17908064 = 2^5.53.10559$	$18017056 = 2^5.79.7127$
*	$18056312 = 2^3.17.103.1289$	$18166888 = 2^3.19.107.1117$

TABLE 1 (Continued)

*	$18194715 = 3^{2}.5.7.11.59.89$	$22240485 = 3^{2}.5.31.107.149$
	$18655744 = 2^{9}.83.439$	$19154336 = 2^{5}.619.967$
*	$20014808 = 2^{3}.11.79.2879$	$21457192 = 2^{3}.47.149.383$
	$20022328 = 2^{3}.17.23.37.173$	$22823432 = 2^{3}.1367.2087$
	$20308995 = 3^{3}.5.7.21491$	$20955645 = 3^{3}.5.17.23.397$
	$21448630 = 2.5.7.131.2339$	$23030090 = 2.5.19.53.2287$
*	$22227075 = 3^{3}.5^{2}.13.17.149$	$24644925 = 3^{3}.5^{2}.29.1259$
	$22249552 = 2^{4}.13.41.2609$	$25325528 = 2^{3}.587.5393$
	$22508145 = 3^{3}.5.11.23.659$	$23111055 = 3^{3}.5.11.79.197$
	$22608632 = 2^{3}.19.23.29.223$	$25775368 = 2^{3}.1439.2239$
	$23358248 = 2^{3}.23.37.47.73$	$25233112 = 2^{3}.37.85247$
	$23389695 = 3^{3}.5.7.53.467$	$25132545 = 3^{3}.5.17.47.233$
*	$23628940 = 2^{2}.5.37^{2}.863$	$27428276 = 2^{2}.17.251.1607$
*	$24472180 = 2^{2}.5.17.167.431$	$30395276 = 2^{2}.47.107.1511$
	$25596544 = 2^{7}.311.643$	$25640096 = 2^{5}.67.11959$
	$25966832 = 2^{4}.29.191.293$	$26529808 = 2^{4}.47.35279$
	$26090325 = 3^{2}.5^{2}.17.19.359$	$26138475 = 3^{2}.5^{2}.11.59.179$
*	$28118032 = 2^{4}.47.139.269$	$28128368 = 2^{4}.59.83.359$
*	$28608424 = 2^{3}.13.139.1979$	$29603576 = 2^{3}.23.349.461$
	$30724694 = 2.7.11.13.103.149$	$32174506 = 2.7.13.17.10399$
	$30830696 = 2^{3}.13.521.569$	$31652704 = 2^{5}.449.2203$
	$31536855 = 3^{2}.5.7.53.1889$	$32148585 = 3^{2}.5.7.102059$
*	$31818952 = 2^{3}.11.41.8819$	$34860248 = 2^{3}.97.167.269$
	$32205616 = 2^{4}.17.167.709$	$34352624 = 2^{4}.2147039$
*	$32642324 = 2^{2}.11.13.149.383$	$35095276 = 2^{2}.17.47.79.139$
	$32685250 = 2.5^{3}.13.89.113$	$34538270 = 2.5.13.379.701$
*	$33501825 = 3^{2}.5^{2}.7.89.239$	$36136575 = 3^{2}.5^{2}.19.79.107$
	$34256222 = 2.7.11.13.71.241$	$35997346 = 2.7.11.23.10163$
*	$34364912 = 2^{4}.43.199.251$	$34380688 = 2^{4}.47.131.349$
	$34765731 = 3^{2}.7.11.13.17.227$	$36939357 = 3^{2}.7.13.23.37.53$
*	$35115795 = 3^{3}.5.11.13.17.107$	$43266285 = 3^{3}.5.53.6047$
	$35361326 = 2.7.11.13.17.1039$	$40117714 = 2.7.13.53.4159$

TABLE 1 (Continued)

	$35373195 = 3^2.5.11.13.23.239$	$40105845 = 3^2.5.13.179.383$
	$35390008 = 2^3.19.23.53.191$	$39259592 = 2^3.71.69119$
	$35472592 = 2^4.43.47.1097$	$36415664 = 2^4.53.42943$
*	$37363095 = 3^2.5.7.11.41.263$	$45663849 = 3^2.7.11.131.503$
*	$37784810 = 2.5.7.539783$	$39944086 = 2.7.13.41.53.101$
	$37848915 = 3^2.5.13.23.29.97$	$39202605 = 3^2.5.13.19.3527$
*	$38400512 = 2^9.179.419$	$38938288 = 2^4.83.109.269$
*	$38637016 = 2^3.11.359.1223$	$40678184 = 2^3.29.271.647$
	$38663950 = 2.5^2.13.17.3499$	$43362050 = 2.5^2.59.14699$
*	$38783992 = 2^3.13.37.10079$	$41654408 = 2^3.47.139.797$
	$38807968 = 2^5.37.73.449$	$40912232 = 2^3.37.89.1553$
*	$43096904 = 2^3.17.41.59.131$	$46715896 = 2^3.53.239.461$
*	$44139856 = 2^4.29.251.379$	$44916944 = 2^4.83.149.227$
	$45263384 = 2^3.17.59.5641$	$46137016 = 2^3.19.433.701$
*	$46237730 = 2.5.7.11^2.53.103$	$61319902 = 2.7.83.113.467$
*	$46271745 = 3^2.5.13.19.23.181$	$49125375 = 3^2.5^3.13.3359$
*	$46521405 = 3^3.5.7.19.2591$	$53011395 = 3^3.5.31.53.239$
*	$46555250 = 2.5^3.7.37.719$	$55880590 = 2.5.103.227.239$
*	$46991890 = 2.5.11.29.14731$	$48471470 = 2.5.19^2.29.463$
*	$48639032 = 2^3.13.29.16127$	$52967368 = 2^3.59.293.383$
	$48641584 = 2^4.29.104831$	$48852176 = 2^4.47.167.389$
*	$49215166 = 2.7.11.13^2.31.61$	$55349570 = 2.5.31.61.2927$
*	$50997596 = 2^2.13.19.71.727$	$51737764 = 2^2.13.23.181.239$
	$52695376 = 2^4.17.151.1283$	$56208368 = 2^4.3513023$
	$56055872 = 2^6.79.11087$	$56598208 = 2^6.383.2309$
*	$56512610 = 2.5.7.11.23.3191$	$75866014 = 2.7^2.774143$
	$56924192 = 2^5.13.193.709$	$64562488 = 2^3.283.28517$
*	$58580540 = 2^2.5.23.347.367$	$70507972 = 2^2.23.521.1471$
*	$59497888 = 2^5.41.101.449$	$61953512 = 2^3.29.97.2753$
*	$63560025 = 3^3.5^2.17.29.191$	$65003175 = 3^3.5^2.23.53.79$
	$63717615 = 3^2.5.13.17.43.149$	$66011985 = 3^2.5.13.19.5939$
	$66595130 = 2.5.7.31.30689$	$74824390 = 2.5.31.59.4091$

TABLE 1 (Concluded)

*	$66854710 = 2.5.13^{3}.17.179$	$71946890 = 2.5.17.83.5099$
	$67729064 = 2^{3}.13.431.1511$	$69439576 = 2^{3}.23.107.3527$
	$67738268 = 2^{2}.13.17.19.37.109$	$79732132 = 2^{2}.19.263.3989$
	$68891992 = 2^{3}.13.23.83.347$	$78437288 = 2^{3}.587.16703$
	$71015260 = 2^{2}.5.23.263.587$	$85458596 = 2^{2}.41.47.11087$
	$71241830 = 2.5.11.19.89.383$	$78057370 = 2.5.17.359.1279$
*	$72958556 = 2^{2}.11.19.197.443$	$74733604 = 2^{2}.11.53.73.439$
	$73032872 = 2^{3}.11.71.11689$	$78469528 = 2^{3}.59.83.2003$
	$74055952 = 2^{4}.23.61.3299$	$78166448 = 2^{4}.197.24799$
*	$74386305 = 3^{2}.5.7.17.29.479$	$87354495 = 3^{2}.5.19.71.1439$
*	$74769345 = 3^{3}.5.7^{2}.89.127$	$82824255 = 3^{3}.5.17.151.239$
	$75171808 = 2^{5}.53.127.349$	$77237792 = 2^{5}.479.5039$
	$75226888 = 2^{3}.11.59.14489$	$81265112 = 2^{3}.53.137.1399$
*	$78088504 = 2^{3}.13.31.53.457$	$88110536 = 2^{3}.167.65951$
*	$78447010 = 2.5.17.19.149.163$	$80960990 = 2.5.11.491.1499$
*	$79324875 = 3^{2}.5^{3}.7^{2}.1439$	$87133365 = 3^{2}.5.11.103.1709$
	$80422335 = 3^{3}.5.7.85103$	$82977345 = 3^{3}.5.11.71.787$
	$82633005 = 3^{2}.5.7.13.17.1187$	$104177619 = 3^{2}.7.13.131.971$
	$83135650 = 2.5^{2}.13.79.1619$	$85603550 = 2.5^{2}.19.251.359$
*	$84521745 = 3^{3}.5.7.11.47.173$	$107908335 = 3^{3}.5.383.2087$
*	$84591405 = 3^{3}.5.17.29.31.41$	$89590995 = 3^{3}.5.13.71.719$
	$86158220 = 2^{2}.5.41.105071$	$99188788 = 2^{2}.23.43.25073$
*	$87998470 = 2.5.7.29.67.647$	$102358010 = 2.5.47.89.2447$
*	$88144630 = 2.5.7^{2}.29.6203$	$102814490 = 2.5.37.269.1033$
	$89477984 = 2^{5}.59.83.571$	$92143456 = 2^{5}.1637.1759$
	$90437150 = 2.5^{2}.19.23.4139$	$94372450 = 2.5^{2}.23.137.599$
*	$91996816 = 2^{4}.29.331.599$	$93259184 = 2^{4}.79.89.829$
	$93837808 = 2^{4}.19.83.3719$	$99899792 = 2^{4}.1399.4463$
	$95629904 = 2^{4}.37.67.2411$	$97580944 = 2^{4}.67.227.401$
	$95791430 = 2.5.7.17.101.797$	$115187002 = 2.7.17.113.4283$
*	$96304845 = 3.5.7^{2}.13.10079$	$96747315 = 3.5.7^{2}.23.59.97$
	$97041735 = 3^{2}.5.7.71.4339$	$97945785 = 3^{2}.5.7.239.1301$

TABLE 2
NEW SOCIABLE GROUPS

(1)	1264460	$2^2.5.17.3719$
	1547860	$2^2.5.193.401$
	1727636	$2^2.521.829$
	1305184	$2^5.40787$
(2)	2115324	$2^2.3^2.67.877$
	3317740	$2^2.5.165887$
	3649556	$2^2.107.8527$
	2797612	$2^2.331.2113$
(3)	2784580	$2^2.5.29.4801$
	3265940	$2^2.5.61.2677$
	3707572	$2^2.11.84263$
	3370604	$2^2.23.36637$
(4)	4938136	$2^3.7.109.809$
	5753864	$2^3.23.31271$
	5504056	$2^3.17.40471$
	5423384	$2^3.53.12791$
(5)	7169104	$2^4.17.26357$
	7538660	$2^2.5.376933$
	8292568	$2^3.59.17569$
	7520432	$2^4.127.3701$
(6)	18048976	$2^4.11.102551$
	20100368	$2^4.919.1367$
	18914992	$2^4.37.89.359$
	19252208	$2^4.1203263$
(7)	18656380	$2^2.5.932819$
	20522060	$2^2.5.13.17.4643$
	28630036	$2^2.19.449.839$
	24289964	$2^2.97.62603$
(8)	28158165	$3^3.5.7.83.359$
	29902635	$3^3.5.7.31643$
	30853845	$3^3.5.11.79.263$
	29971755	$3^3.5.11.20183$
(9)	46722700	$2^2.5^2.47.9941$
	56833172	$2^2.11.53.24371$
	53718220	$2^2.5.2685911$
	59090084	$2^2.43.343547$

Note Added. After the first version of this paper was submitted to *Math. Comp.*, I was informed that Paul Bratley, John McKay, and Fred Lunnon had independently computed the amicable pairs from 10^7 to 10^8. Their 128 pairs agree exactly with mine.

Results on Sociable Numbers. Until now only two groups of sociable numbers were known, respectively of order 5 and 28; both were found by Poulet [8]. I have made an exhaustive search for sociable groups of order $t \leqq 10$ of which the lesser number is smaller than 6.10^7. This search has yielded 9 new groups, which interestingly enough are all of order 4. They are given in Table 2.

This relative abundance of order 4 sociables compared with other orders is rather surprising and calls for some comments.

Let us say that a sociable group is a *regular* group of order t if it is of the form $(a \cdot n_1, \cdots, a \cdot n_t)$ with each n_i prime to a for $1 \leqq i \leqq t$ and $n_1, \ldots, n_t$ have no common factor. Then a theorem of Dickson [10], states that there are no regular groups of odd order > 1. On the other hand, of the 236 amicable pairs up to 10^8, 193 are regular, and of the 9 sociables of order 4, 7 are regular. Regular groups thus seem to form the large majority of groups of even order 2 and 4, so Dickson's theorem can explain, at least partly, why only one group has been found of odd order > 1. It does not explain why no groups of order 6, 8 or 10 have been found.

Results on Unbounded Sequences. It has been conjectured by Catalan (see revision by Dickson [10]) that the aliquot series of n is never unbounded. It is known to be bounded for $2 \leqq n \leqq 275$. The smallest n for which the behavior is not known is 276. G. A. Paxson [9] has calculated 67 terms of this sequence. I have extended this to 119 terms and found:

$$a_{118}(276) = 2133148752623068133100.$$

Conclusion. From these results a number of conjectures can be made.

Let $A(x)$ be the number of amicable pairs of which the smaller number is less than x; then empirically one can conjecture:

Conjecture 1. There exists $\beta > 0$ such that

$$\text{Log } A(x) \sim \beta \cdot \text{Log}(x).$$

This conjecture of course implies the as yet unknown fact that there exists an infinity of amicable pairs.

From Table 1 and preceding tables a least square method gives

$$\beta = 0.29 \ldots.$$

A heuristic computation of β would be welcome.

Conjecture 2. There exists an infinity of sociable groups of order 4.

This is a particular case of a general conjecture of Erdös [11]. Furthermore in the same paper Erdös states that the density of sociable groups of any order is 0. Combining this with Catalan's conjecture as revised by Dickson one obtains:

Conjecture 3. For almost all n (i.e. with density 1) the associated sequence converges.

These conjectures seem very difficult to prove.

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