REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the indexing system printed in Volume 22, Number 101, January 1968, page 212.

20[2.00, 3, 10, 11, 12, 13.05, 13.15, 13.30].—R. SAUER & I. SZABÓ, Editors, *Mathematische Hilfsmittel des Ingenieurs*. III, Authors: T. P. ANGELITCH, G. AUMANN, F. L. BAUER, R. BULIRSCH, H. P. KÜNZI, H. RUTISHAUSER, K. SAMELSON, R. SAUER, J. STOER, Springer-Verlag, Berlin, 1968, xviv + 534 pp., 24 cm. Price $24.50.

This is the third volume of a projected four-volume treatise. Volume I has previously appeared (see Review 3, this Journal, v. 23, 1969, pp. 208–209). To quote from the Editors' Preface, the work as a whole "is designed to acquaint the engineer with the modern state of mathematics as it pertains to theories and methods which are, or promise to become, of significance in engineering. . . . [The work] is more than a compendium of formulas in the customary sense. In each of the disciplines covered, it not only brings the necessary formal apparatus, but also the basic definitions, theorems, and methods, in a presentation which takes account of the physically-geometrically oriented mentality of the engineer. That is, concepts introduced in definitions are interpreted intuitively, whenever possible, and a constant effort is made to motivate the reader for the concepts introduced. The understanding of theorems and methods is facilitated by use of examples and plausibility arguments. Proofs are given only in those cases where they are essential for the understanding of a theorem or a method. Through appropriate references to text books, however, the reader is put in a position to orient himself in each case on pertinent proofs. . . . Although [this work] responds primarily to the needs of engineers, it can profitably be used also by natural scientists, especially physicists and mathematicians."

The present volume contains six chapters, labeled F through K. Chapter F (85 pages), by F. L. Bauer and J. Stoer, is devoted to algebra and presents basic notions and theorems in algebraic structures (semigroups, groups, rings, fields and skew fields), a concise exposition of linear algebra and normed vector spaces, and has also a section on localization theorems for zeros of polynomials and eigenvalues.

Chapter G (146 pages), devoted to geometry, has two parts. Part I, by R. Sauer, deals with selected topics of geometry: affine and projective geometry, conic sections and quadratic surfaces, nomography, spherical trigonometry, vector algebra and vector analysis, differential geometry of curves and surfaces with applications to kinematics, and curvilinear coordinate systems. Part II, by T. P. Angelitch, brings a detailed exposition of tensor calculus, including applications to classical mechanics, heat conduction, electrodynamics, and continuum mechanics.

Chapter H (88 pages), by R. Bulirsch and H. Rutishauser, has interpolation and numerical quadrature as its topic. It contains a comprehensive discussion, frequently supplemented by ALGOL procedures, of polynomial and rational interpolation procedures, interpolation and smoothing by spline functions, and polynomial interpolation in several variables. This is followed by an exposition of selected quadrature schemes, including the Romberg scheme, Newton-Cotes and Gaussian quadrature, and other miscellaneous integration formulas.

Chapter I (127 pages), concerned with the approximation of functions, is again in two parts. Part I, by G. Aumann, outlines the mathematical foundations of approximation theory. Part II, by R. Bulirsch and J. Stoer, addresses itself to the effective computation of functions on digital computers. Among the topics treated are Chebyshev expansion, the use of continued fractions, computation of elliptic functions by Bartky's transformation. Fourier analysis, including the Cooley-Tukey algorithm, and the recursive computation of cylinder functions. A number of ALGOL procedures are included.

Chapter J (51 pages), by H. P. Künzi, treats linear and nonlinear optimization problems. There is a brief outline of the mathematical theory of linear optimization, which is followed by a description of constructive solution algorithms, including Dantzig's simplex method, Gomory's integer programming algorithm, and the author's duoplex algorithm. On nonlinear problems one finds the Kuhn-Tucker theorem for convex problems, and Beale's algorithm for quadratic problems.

The final Chapter K (19 pages), by K. Samelson, starts with an intuitive introduction to the concepts of model, algorithm, and program, and continues to survey the organization of stored program digital computers and problem-oriented programming languages.

The aims set by the editors have been admirably achieved in this volume, and one anxiously looks forward to the appearance of the remaining two volumes.

W. G.

21[2.05].—C. T. Fike, *Computer Evaluation of Mathematical Functions*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1968, xii + 227 pp., 24 cm. Price $10.50

The title of this volume is somewhat misleading inasmuch as the mathematical functions considered are largely elementary functions, and then only with real arguments. Accordingly, the methods of evaluation are those of polynomial and rational approximation (plus Newton's iteration in the case of the square- and cube-root). Within these restrictions, however, the author has given us an account which is eminently readable, sound in mathematical and computational detail, and rich in illustrative examples and cogent remarks. The treatment is thoroughly up-to-date and well documented by references, not only to the research literature, but also to manufacturer-supplied program libraries. The text can be highly recommended for reference use and for supplementary reading in a numerical analysis course at the junior-senior level.

The territory covered is well delineated by the chapter headings: 1. Error in Function Evaluation Computations; 2. Square-Root and Cube-Root Evaluation; 3. Reducing the Argument Range; 4. Polynomial Evaluation Methods; 5. Minimax Polynomial Approximations; 6. Chebyshev Polynomials and Chebyshev Series; 7. Various Polynomial Approximation Methods; 8. Rational-Function Evaluation Methods; 9. Minimax Rational Approximations; 10. Various Rational Approximation Methods; 11. Asymptotic Expansions. Each chapter is followed by a bibliography and a set of exercises.

W. G.

22[2.05].—THEODORE J. RIVLIN, *An Introduction to the Approximation of Functions*, Blaisdell Publishing Co., Waltham, Mass., 1969, viii + 150 pp., 24 cm. Price $7.50.

This is a well written, enjoyable book. It requires only a good undergraduate mathematics program as background. The author is quite successful in his compromise of restricting the discussion to concrete interpolation and approximation procedures, without losing sight of the basic mathematical ideas, which can be carried over to more general situations. It is an elementary book in the best sense of the word.

The main emphasis of the book is on approximation by polynomials and by piecewise polynomials. What follows is a short description of the various chapters.

A short introduction gives the abstract existence theory, discusses the role of uniform convexity and introduces the necessary spaces.

Chapter 1. "Uniform Approximation," is a skillful presentation of Weierstrass' theorem, Jackson's theorems and the Chebyshev characterization of the best approximation. The corresponding problem on finite point sets is discussed as well as two numerical procedures.

Chapter 2. "Least Squares Approximation," contains a treatment of approximation with orthonormal polynomials on a bounded interval as well as on a finite set of points. Also included is a discussion of the effectiveness, as a uniform approximation, of least squares approximation.

Chapter 3. This chapter develops a theory of least-first-power approximation on intervals and finite point sets. It is shown that the solution of the discrete problem converges to that of the continuous one under appropriate conditions. The solution of the discrete problem by linear programming is discussed.

Chapter 4. Here polynomial and spline interpolation are treated. The chapter starts with an illuminating discussion of Lagrange interpolation, the convergence problem, and how the convergence depends on the location of the nodes. Next follows a discussion of the effectiveness of interpolation polynomials in the least-square and least-first-power sense. The end of the chapter discusses cubic splines. Existence of a cubic spline interpolant is proved as well as extremal properties. The use of splines for least-square and uniform approximation is discussed and an error estimate is given in the uniform norm.

Chapter 5. This last chapter discusses the characterization of rational approximation and interpolation on finite intervals and finite point sets. The author also treats numerical procedures for these problems.

A series of exercises increases the usefulness of this excellent textbook.

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23[2.05, 2.10, 2.55, 4, 5, 13.05].—L. COLLATZ, G. MEINARDUS & H. UNGER, Editors, *Numerische Mathematik, Differentialgleichungen, Approximationstheorie*, Birkhäuser Verlag, Basel, 1968, 401 pp., 25 cm. Price SFR 48—.

Functional analysis, as a unifying agent, has become of increasing significance in many disciplines of mathematics, numerical mathematics being no exception. The desire to elucidate the interactions between functional analysis and numerical mathematics, particularly the numerical treatment of differential equations and approximation theory, led in 1966 to two working conferences being held at the Mathematics Research Institute Oberwolfach (Black Forest). The first conference, under the direction of L. Collatz and H. Unger, was devoted to differential equations, the second, under the direction of L. Collatz and G. Meinardus, to numerical analysis, especially approximation theory. The present volume contains the proceedings of these conferences. The contributions vary in length and content, ranging from short abstracts to preliminary research reports, expository articles, and complete original research papers. The authors and their titles are listed below:

Conference on the Numerical Treatment of Differential Equations, June 20–25, 1966

H. Amann, Monte-Carlo Methoden zur Lösung elliptischer Randwertprobleme
R. Ansorge, Zur Frage der Verallgemeinerung des Äquivalenzsatzes von P. D. Lax
I. Babuška, Optimierungsprobleme numerischer Methoden
G. Bruhn, Ein Charakteristikenverfahren für instationäre Strömungen entlang bewegter Wände
B. Dejon, Stabilitätskriterien in Abhängigkeit von den Normen für die Startwerte
S. Filippi, Neue Lie-Reihen-Methode
F. Krückeberg, Defekterfassung bei gewöhnlichen und partiellen Differentialgleichungen
K. Nickel und P. Rieder, Ein neues Runge-Kutta-ähnliches Verfahren
J. Nitsche, Zur Konvergenz des Ritzschen Verfahrens und der Fehlerquadratmethode I
G. Opitz, Einheitliche Herleitung einer umfassenden Klasse von Interpolationsformeln und Anwendung auf die genäherte Integration von gewöhnlichen Differentialgleichungen
P. Rózsa, Ein Rekursionsverfahren zur Lösung linearer Differentialgleichungssysteme mit singulären Koeffizientenmatrizen
J. W. Schmidt und H. Schönheinz, Fehlerschranken für die genäherte Lösung von Rand- und Eigenwertaufgaben bei gewöhnlichen Differentialgleichungen durch Differenzenverfahren
H. Schwermer, Zur Fehlererfassung bei der numerischen Integration von gewöhnlichen Differentialgleichungssystemen erster Ordung mit speziellen Zweipunktverfahren
H. J. Stetter, Stabilitätsbereiche bei Diskretisierungsverfahren für Systeme gewöhnlicher Differentialgleichungen
W. Törnig, Über Konvergenzbereiche von Differenzapproximationen bei quasilinearen hyperbolischen Anfangswertproblemen
W. Walter, Wärmeleitung in Systemen mit mehreren Komponenten
W. Wendland, Zur numerischen Behandlung der Randwertaufgaben für elliptische Systeme
W. Wetterling, Lösungsschranken beim Differenzenverfahren zur Potentialgleichung

Conference on Numerical Analysis, especially Approximation Theory, November 13–19, 1966

J. Blatter, Zur stetigen Abhängigkeit der Menge der besten Approximierenden eines Elementes in einem normierten reellen Vektorraum
B. Brosowski, Rationale Tschebyscheff-Approximation differenzierbarer Funktionen
E. W. Cheney and A. A. Goldstein, A Note on Nonlinear Approximation Theory
E. Gröbner, Approximationen durch Umordnungen von Lie-Reihen
D. Henze, Über nichtlineare Approximationen in linearen normierten Räumen
W. Krabs, Über ein Kriterium von Kolmogoroff bei der Approximation von Funktionen
J. Meinguet, Optimal Approximation and Error Bounds in Normed Spaces
K. Nickel, Anwendungen einer Fehlerschranken-Arithmetik
R. Nicolovius, Extrapolation bei monoton zerlegbaren Operatoren
M. J. D. Powell, On Best L_2 Spline Approximations
J. Schröder, Monotonie-Aussagen bei quasilinearen elliptischen Differentialgleichungen und anderen Problemen
F. Schurer and F. W. Steutel, Approximation with Singular Integrals of the Jackson Type
P. C. Sikkema, Über Potenzen von verallgemeinerten Bernsteinoperatoren
J. J. Sopka, Über verallgemeinerte numerische Integrationen
H. Werner, Diskretisierung bei Tschebyscheff-Approximation mit verallgemeinerten rationalen Funktionen
W. Wetterling, Lösungsschranken bei elliptischen Differentialgleichungen.

W. G.

24[2.10].—ROBERT PIESSENS, *Gaussian Quadrature Formulas for the Integration of Oscillating Functions*, 2 pages of tables and 1 page of explanation, reproduced on the microfiche card attached to this issue. Review taken from author's explanation.

A table of weights and abscissas (to sixteen significant figures) for the $2n$-point Gaussian integration formula

$$\int_{-\pi}^{\pi} f(x) \sin x \, dx = \sum_{j=1}^{n} [w_j f(x_j) - w_j f(-x_j)], \qquad 0 < x_j < \pi,$$

is given for $n = 1(1)9$. The abscissas x_j and weights w_j are given by

$$x_j = \sqrt{u_j}, \qquad w_j = b_j/x_j, \qquad j = 1, 2, \ldots, n,$$

where u_j and b_j are the abscissas and weights of the n-point Gaussian quadrature formula

$$\int_0^{\pi^2} \frac{1}{2} \sin \sqrt{u}\, g(u)\, du = \sum_{j=1}^{n} b_j g(u_j).$$

The author chose the method of Anderson [1], to compute the u_j and b_j. The calculations were performed on the IBM 1620 of the Computing Center of the University of Leuven.

E. I.

1. D. G. Anderson, "Gaussian quadrature formulae for $\int_0^1 -\ln(x) f(x)\, dx$," *Math. Comp.*, v. 19, 1965, pp. 477–481.

25[2.10, 7].—David M. Bishop, *Evaluation of Certain Integrals of Reduced Modified Bessel Functions of the Second Kind*, ms. of 6 typewritten pages (undated) deposited in the UMT file.

The main table in this manuscript consists of 8S floating-point values of the integral

$$I_n(\mu) = \int_0^{\infty} [x^{\mu} K_{\mu}(x)]^4 x^n \, dx$$

for $n = 0(1)4$ and $\mu = 0.1(0.1)5$. The numerical integration was accomplished by means of Gauss-Laguerre quadrature, using 32 points.

When μ is half an odd integer (and n is a nonnegative integer) the value of $I_n(\mu)$ is expressible as a rational multiple of π^2; accordingly, the author presents in a supplementary table these rational coefficients (with their decimal equivalents to 8S) for $n = 0(1)4$ and $\mu = 0.5(1)2.5$. From a comparison of corresponding entries in the two tables the author concludes that the main table is accurate to at least 7S. (This reviewer noted one instance of accuracy to 6S, corresponding to $n = 0$, $\mu = 1.5$.)

The author states that this integral appears in certain atomic and molecular calculations where the electronic wave function is chosen as a combination of Bessel functions.

J. W. W.

26[2.30, 2.45, 8, 9, 12].—Donald E. Knuth, *The Art of Computer Programming*, Vol. II: *Seminumerical Algorithms*, Addison-Wesley Publishing Co., Reading, Mass., 1969, xi + 624 pp., 25 cm. Price $18.50.

In this impressive sequel to the first volume [1] of his projected seven-volume series on the art of computer programming the author considers those aspects of computer programming most closely related to classical mathematics, numerical analysis, and statistics.

As concisely stated in the preface, the problem discussed here is "to find the best ways to make computers deal with numbers".

The present volume consists of two additional chapters in the complete work; namely, Chapter 3, entitled "Random Numbers", and Chapter 4, entitled "Arithmetic".

The introduction to Chapter 3 contains an enumeration of various areas in which random numbers find useful application; of these, that of primary interest in this book is the testing of the effectiveness of computer algorithms by means of such data. The well-known "middle-square" method of generating "random" numbers is carefully examined, and the conclusion is reached that such numbers should not be generated by a method chosen at random. Superior random-number generators, considered next, include the linear congruential method, the quadratic method of Coveyou, the Fibonacci recurrence, and the combination of such generators.

Statistical tests for randomness are treated at length in Section 3.3 (pp. 34–100). These include general tests (the chi-square test and the Kolmogorov-Smirnov test), empirical tests (frequency, serial, gap, poker, coupon-collector's, permutation, and run tests), and theoretical tests in connection with linear congruential sequences. The important spectral test, which was developed in 1965 by Coveyou and MacPherson, is discussed in considerable detail (pp. 82–100).

Next we find in Section 3.4 a useful summarization of the best known procedures for producing numbers from various standard statistical distributions, followed by a discussion of random sampling and shuffling (with algorithms attributed to Ulam and Durstenfeld).

In Section 3.5 (pp. 127–151), which includes historical and bibliographic information, the author considers the basic question as to what constitutes a random sequence. On p. 142 he formulates a definition of such a sequence, which he believes satisfies all reasonable philosophical requirements.

A useful summary of this chapter presents a procedure for obtaining the "nicest" and "simplest" random-number generator and concludes with a brief discussion of additional references for supplementary reading.

Chapter 4 contains detailed analyses of algorithms for performing arithmetic operations on a variety of quantities such as floating-point numbers, multiple-precision numbers, rational numbers, polynomials, and power series. Related topics include radix conversion, factorization of integers, and evaluation of polynomials.

One of the most entertaining and informative sections in this book, in the opinion of the reviewer, is the first one of this chapter. Positional number systems and their historical development are here treated, with appropriate attention to the influence of electronic digital computers. In particular, unusual number systems such as the factorial system and those involving complex as well as negative bases are also considered.

In Section 4.2 single- and double-precision floating-point calculations are successively treated, with particular attention to the inherent accuracy of such calculations performed on electronic computers. An important section describes the common pitfalls encountered in the preparation of floating-point routines. Illustrative programs are written in MIX language, a description of which appears in Appendix A (pp. 565–595), as reproduced from Volume 1 (pp. 120–152).

The general subject of multiple-precision arithmetic is then considered. Algorithms, accompanied by MIX programs, are stated for the four fundamental arithmetic operations and formulas are derived for the corresponding running time in computer cycles.

An alternative approach to performing arithmetical operations on large integers, namely modular arithmetic, is next described and the application of this method to the exact solution of linear equations with rational coefficients is mentioned.

The challenging question, "How fast can we multiply?" is discussed at great length. Apparently, a faster method than the familiar one of order n^2 for multiplying multiple-precision numbers was not discovered until 1962. The improved method of Toom & Cook, especially adapted for electronic computers, and the modular method of Schönhage are both described in detail, including appropriate algorithms.

In Section 4.4 radix conversion is briefly discussed, including four basic methods and illustrative examples involving single- and multiple-precision conversion, floating-point conversion, and hand calculation.

The next topic of wide importance treated here is that of rational arithmetic in Section 4.5 (pp. 290–360). The fundamental algorithm is that of Euclid for finding the greatest common divisor of two integers. The intimate connection between this algorithm and regular continued fractions leads to a discussion of such details as Lamé's theory and the research of Lévy.

The author devotes nearly 22 pages to the perennial problem of factoring integers. The methods described include division by successive primes, Fermat's method, use of sieves, and Legendre's method. Tests of primality, including the Lucas-Lehmer test, are also described. An up-to-date account of the search for Mersenne primes is included. Number theorists should find the list of special prime numbers on p. 355 particularly interesting and useful.

The following discussion of polynomial arithmetic in Section 4.6 leads to a study of addition chains (pp. 398–422) which are encountered in a search for the most economical way to compute x^n by multiplication. The problem of the efficient evaluation of polynomials leads to discussions of such procedures as Horner's rule, Yate's method, Lagrange's and Newton's interpolation formulas, and Pan's method. Finally, the concept of addition chains is extended to polynomial chains in connection with the minimization of the number of arithmetic operations for evaluating polynomials of various degrees.

The concluding section of this chapter deals with the manipulation of power series. The transformation considered in greatest detail here is that of reversion of series, and three algorithms are presented for that purpose.

The wealth of information in this volume is supplemented by more than 650 exercises of graduated difficulty, ranging from trivial problems to unsolved research questions. As in Volume I, a large part of this book (pp. 452–564) is devoted to the answers to the great majority of these exercises.

Tables of important constants to 40 decimal places and 44 octal places, in addition to the first 25 harmonic, Bernoulli, and Fibonacci numbers, are presented in Appendix B (pp. 596–599). A useful index to notations appears in Appendix C (pp. 600–604).

This exceptionally scholarly, informative book is a worthy companion to the first volume of this series. It should be read by all persons aspiring to become experts in the fields of numerical analysis and computer programming, and it certainly can be studied with profit by students in both pure mathematics and statistics.

J. W. W.

1. DONALD E. KNUTH, *The Art of Computer Programming*, Vol. I: *Fundamental Algorithms*, Addison-Wesley Publishing Co., Reading, Mass., 1968. (See *Math. Comp.*, v. 23, 1969, pp. 447–450, RMT **18.**)

27[3].—GEORGE E. FORSYTHE & CLEVE B. MOLER, *Computer Solution of Linear Algebraic Systems*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1967, xi + 148 pp., 24 cm. Price $6.75.

This is an excellent brief introduction to the subject, written by two experts with considerable experience. The senior author, in fact, is one of the pioneers.

Very little is presupposed on the part of the reader. The necessary theoretical background is developed in an elementary fashion, and detailed algorithms are spelled out and analyzed. For the beginner, and even for those who already have some experience, this book is a must.

A. S. H.

28[3, 10].—MORRIS NEWMAN, *Matrix Representations of Groups*, Applied Mathematics Series No. 60, National Bureau of Standards, Washington, D. C., 1968, 79 pp., 26 cm. Price $0.60.

This monograph develops the theory of representations of groups in terms of finite dimensional matrices over the field of complex numbers, with strong emphasis on the representation theory of finite groups. It avoids algebraic machinery outside of matrix theory as far as possible, trying successfully to give proofs which are both elementary and simple. Appendices deal with the elements of the theory of algebraic numbers (needed, e.g., for proving the solvability of groups whose order is divisible by not more than two distinct prime numbers) and specifically with the roots of unity. An interesting proof of the irreducibility of the cyclotomic polynomials is included.

Except for the monograph by Martin Burrow [*Representation Theory of Finite Groups*, 185 pp., Academic Press, New York, 1965] (which goes farther and develops and uses more advanced algebraic tools), there seems to exist no book of comparable size in English which gives the same amount of information; the book by Curtis and Reiner [*Representation Theory of Finite Groups and Associative Algebras*, 685 pp., Interscience, New York, 1962] is much larger, and the book by Marshall Hall [*The Theory of Groups*, 434 pp., Macmillan, New York, 1959], which contains representation theory as a chapter, covers many other parts of group theory as well.

There is no doubt that the present monograph will be useful for many purposes and to many readers. In particular, the explicit construction of a full set of irreducible representations for some finite groups may be welcome to many users.

The reviewer found the book very clear locally, but less so globally. It follows, of course, from the proved results, that all finite dimensional representations of a finite group are equivalent to a matrix representation which is composed in an obvious manner of a finite number of irreducible representations, and that a knowledge of the character function of any representation describes its composition completely. But statements of this and of a similar type (e.g., concerning the importance of the regular representation) which could serve as landmarks for the reader, in what amounts to a very wide range of facts, are not, or certainly not conspicuously, displayed.

There are, of course, the nearly unavoidable small inaccuracies. Theorem 12, p. 68 uses the term "algebraically closed" for the ring of algebraic integers—whereas not even all linear equations with coefficients in this ring have a solution that is in the ring.

Also, on p. 4, it would be better to say that there are groups without finite dimensional representations (in spite of the author's definition of a representation at the top of p. 4 which excludes infinite dimensional representations if n is tacitly assumed to be finite). After all, there exists a large and growing literature on infinite dimensional representations of some groups. On p. 12, a reference [G. Higman, B. H. Neumann, H. Neumann, *J. London Math. Soc.*, v. 24, 1949, pp. 247–254] could be given for the construction of infinite groups with two conjugacy classes.

Finally, there are some questions of method. In proving Theorem 3, p. 64, the author refers to van der Waerden for the theory of symmetric functions. But the proof given in van der Waerden for the same theorem just uses a little linear algebra—and would have fitted nicely into the text. Also, it is not clear to the reviewer why the author, after abstaining from using ideals (and, e.g., Wedderburn's Theorem) in the main part of the text, proves the unique factorization into prime ideals for the integers of algebraic number fields in the Appendix.

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29[3, 12].—T. J. DEKKER & W. HOFFMAN, *ALGOL* 60 *Procedures in Numerical Algebra, Part* II, Mathematical Centre Tracts 23, Mathematisch Centrum, Amsterdam, 1968, 95 pp., 24 cm. Price $3.00.

The authors present the programs which they have developed at the Mathematical Center in Amsterdam for calculating eigenvalues and eigenvectors of real matrices which may be stored in high speed memory. Their procedures make use of the basic subroutines for matrix and vector operations which the authors presented in Part I.

This is an excellent collection of programs developed with understanding and loving care, fully comparable with the Handbook series of Numerische Mathematik. Documentation is thorough. To each page of code, there is at least a page of description. However, this book does not claim to be a textbook on these numerical methods. Familiarity with the subject matter is assumed in the descriptions, whose purpose is to present the all important programming details which can make or break a procedure.

Each chapter begins with a survey of its subdivisions. Each subdivision corresponds to a particular procedure, explains the numerical method and gives the necessary details.

The first chapter (Chapter 23 of the series) concerns symmetric matrices. These are reduced to tridiagonal form by Householder's method. The pth step introduces zeros into the $(n - p + 1)$th row and column. What was the reason for reversing the usual order, one wonders? The pth step is skipped if the elements to be annihilated are smaller than the machine precision times the norm of the matrix. This reviewer has serious reservations about this decision. For example, the off diagonal 1's would be ignored in the matrix

$$\begin{pmatrix} 10^{12} & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

This produces answers with errors which are large relative to the true values and negligible compared with the matrix norm. The point, however, is that there is no difficulty in producing answers which are correct and it seems a pity not to do this when possible. The solution is to skip a step only when the elements to be annihilated are already zero. Occasionally, unnecessary transformations will be performed, but they will do no harm.

Two methods are given for calculating the eigensystems of tridiagonal matrices: iterated linear interpolation and the QR transformation.

The linear interpolation routine, called zeroin, is a clever combination of the secant method, the method of regula falsi, and bisection. Yet it takes only 18 lines of ALGOL to define. It is a fine example of the art of coding numerical methods. It avoids the pitfalls which beset earlier implementations.

The characteristic polynomial is evaluated by triangular factorization of the tridiagonal matrix $T - xI$ (without interchanges). This technique avoids the under/overflow problems which can plague the Sturm sequence of principal minors. Eigenvectors are found by inverse iteration. Multiple (or very close) eigenvalues are not tolerated (à la Wilkinson) and a Gram-Schmidt process is invoked, if necessary, to force orthogonality on the eigenvectors.

Two versions of the QR transformation for tridiagonals are given: a standard version which uses plane rotations and also gradually builds up the matrix of eigenvectors and, when only eigenvalues are required, the square-root-free version due to Ortega and Kaiser.

Chapter 24 gives procedures for calculating the eigensystems of real matrices with the aid of the QR transformation. The matrices are first balanced and then converted to upper Hessenberg form by stabilized elementary transformations. Both the single and double QR iterations are available for reducing the Hessenberg matrix to block triangular form. The former is used on matrices whose eigenvalues are known to be real. Eigenvectors may be found by using inverse iteration or by saving the product of the Q-matrices and finding the eigenvectors of the final matrix of the QR sequence directly. The latter method is only given for matrices with real eigenvectors. The row eigenvectors are not calculated explicitly. They may be found by inverting the matrix of column eigenvectors produced by these procedures.

One might well wonder whether the problem of computing eigenvalues is now essentially solved. The answer is no (or, at least, not quite). We do not yet understand the convergence properties well enough and we do not know completely satisfactory and economical tolerances and termination criteria for iterations. Sometimes our answers suffer a quite unneccessary lack of accuracy. On the other hand, the publication of these algorithms demonstrates the tremendous progress which has been made during the last fifteen years. For example, no multiple precision is needed anywhere in this booklet. The extent of the details which had to be mastered by the authors is indicated by the fact that nearly 100 pages are required to describe the programs.

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30[3, 13, 15].—LOUIS A. PIPES & SHAHEN A. HOVANESSIAN, *Matrix Computer Methods in Engineering*, John Wiley & Sons, Inc., New York, 1969, xi + 333 pp., 24 cm. Price $12.95.

This book is written at the junior-senior level; it includes a number of numerical examples and exercises, as well as a number of programs in FORTRAN and in BASIC; about half a dozen references are listed in each chapter. The first four chapters, about 140 pages, are concerned with general theory and general numerical techniques; the remaining five chapters, about 180 pages, take up a variety of applications: electricity, time-frequency domain, vibrations (conservative and nonconservative systems), and structures. There is a five-page index.

The objectives of a course that would use this as a text are hard to imagine. Perhaps it could give the students some feeling for the range of applicability of matrices. But according to the title, the book has to do with methods, and, as already mentioned, there are a fair number of actual programs. But the theory is minimal, and the basic computational techniques described are largely obsolete (barring the inevitable Gaussian elimination) except for very small matrices, say, of order four or five.

The power method is described and illustrated in Chapter 3. Justification is presented in Chapter 7. The power method for roots of smallest modulus is given also in Chapter 3, using the explicit inverse. The Danilevski method for obtaining the characteristic polynomial is given, and so is a version of the method of Le Verrier, attributed to Bocher. Nothing is said about rounding errors, and the name of Wilkinson nowhere occurs. Nothing is said about the Hessenberg or the tridiagonal form. And naturally nothing is said about the use of the inverse power method to refine an approximate root. Among the references, the most recent one is the very poor English translation of the first edition of Faddeev and Faddeeva, the translation having the date 1963, whereas the original appeared in 1960. And yet, in seeing the various programs, the student could easily get the idea that this is the last word.

A. S. H.

31[4, 5, 6, 13.05, 13.15].—S. G. MIKHILIN & K. L. SMOLITSKIY, *Approximate Methods for Solution of Differential and Integral Equations*, American Elsevier Publishing Co., New York, 1967, vii + 308 pp., 24 cm. Price $14.00.

This is a translation from the Russian of a reference book published in 1965 by Nauka Press, Moscow, in its series "Spravočnaja Matematičeskaja Biblioteka." The work gives an excellent exposition, on an advanced level, of the most important approximate methods for solving boundary-value problems for differential equations, both ordinary and partial. It also considers the numerical solution of Fredholm and

Volterra integral equations, and singular integral equations. While detailed derivations and proofs are omitted, the pertinent theories are set forth with great clarity. A distinctive feature of the work are the numerous examples inserted throughout the text, drawing from a variety of topics in mathematical physics and engineering, and richly illustrating the theoretical results presented. There are also frequent references to the original literature.

The considerable scope of the work can best be seen from the following table of contents.

The translation by Scripta Technica, Inc., under the editorial supervision of Robert E. Kalaba, is carried out with competence. The editor is to be commended for making this important work accessible to the English-speaking mathematical community.

W. G.

32[4, 13.05].—FRED BRAUER & JOHN A. NOHEL, *Ordinary Differential Equations, A First Course*, W. A. Benjamin, Inc., New York, 1967, xvi + 457 pp., 24 cm. Price $12.75.

This is a text for an introductory course in ordinary differential equations, designed for students with no previous experience in the subject. Chapter 1 uses some typical mechanical systems to motivate the study of differential equations. Chapter 2 brings elementary methods of solution, including Euler's method as an example of an approximate method. Chapter 3 is devoted to the general theory of linear differential equations, while Chapter 4 concentrates on linear second-order equations, and presents a detailed treatment of solutions in power series, including a study of Bessel's equation, and an introduction to asymptotic expansions (arising naturally in connection with irregular singular points). Chapter 5 discusses boundary value problems, proceeding from elementary examples to general Sturm-Liouville problems. Up to this point, the material can be mastered by students with no background in linear algebra. A knowledge of the elements of vector algebra will be needed for Chapter 6, which deals with systems of linear and nonlinear differential equations, and briefly with the qualitative theory of autonomous systems. The basic existence and uniqueness theorems, as well as theorems on the continuous dependence on initial data and parameters are proved in Chapter 7, both for single differential equations and systems of differential equations. Chapter 8 gives an introduction to numerical methods with a good discussion of error accumulation. Chapter 9, finally, is on Laplace transforms and their use for solving initial value problems for linear differential equations. There are numerous exercises, some scattered throughout the chapters, others collected at the end of each chapter.

While maintaining a high standard of mathematical exposition, the authors have succeeded in blending theory with applications so as to impart to the student an appreciation not only of the mathematical coherence of the subject, but also of its usefulness as a tool to "explain and help him understand various physical phenomena in the physical world".

W. G.

33[5].—DONALD GREENSPAN, *Lectures on the Numerical Solution of Linear, Singular, and Nonlinear Differential Equations*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1968, 185 pp., 26 cm. Price $6.95.

This book is a survey of numerical methods for the solution of differential equations, based on the author's lectures at summer conferences at the University of Michigan. According to the preface, "Scientists and technologists should be able to determine easily from the text what the latest methods are and whether those methods apply to their problems". There are 486 references that "will enable teachers to adapt the material for classroom presentation".

The reviewer would hesitate very much to use this book as a textbook or to suggest it as reading for applied scientists asking for advice on numerical methods. The material is quite specialized, and only a few methods are discussed. No proofs are given. In the discussion of elliptic problems, the problem of convergence and accuracy is not even mentioned, and the reader is left without any guidance as to why one difference approximation might be preferable to another. Instead, a lot of space is taken up by a detailed and repetitive discussion of how the replacement of differential operators by finite differences leads to systems of algebraic equations. Frequently, numerical values for the coefficients are given for some specific mesh-size.

The only method discussed for hyperbolic and parabolic problems is one invented by the author, which changes the problem into a boundary value problem. This is odd in a book which claims to be a survey. For several decades, the more conventional marching procedures have been used, often with great success, by an enormous number of people. The basic algorithmic ideas behind these methods, as well as a simplified stability theory, could have been presented easily, even to an audience which is not very sophisticated mathematically.

The last chapter deals with the author's method for the steady state Navier-Stokes problem. The author describes a series of numerical experiments, using only 81 interior meshpoints for Reynold's numbers up to 10^5. There is no discussion of accuracy; in fact, it should be obvious that the flow described by the discrete model in such a case has only a formal connection with the differential equations. Because of the fact that such flows cannot be described with so few parameters, the treatment of these calculations should have been omitted or else put into proper perspective.

The author is well known as a master of the very sophisticated art of obtaining numerical solutions to difficult applied problems. But this book does not fulfill the promise indicated in the preface, because it concentrates on his own work and neglects too many important methods.

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34[7].—B. S. Berger & H. McAllister, *A Table of the Modified Bessel Functions* $K_n(x)$ *and* $I_n(x)$ *to at Least* 60S *for* $n = 0, 1$, *and* $x = 1, 2, \ldots, 40$, ms. of 4 typewritten sheets + 8 computer sheets (reduced) deposited in UMT file. (Copies also obtainable from Professor Berger, Department of Mechanical Engineering, University of Maryland, College Park, Md. 20742.)

Assisted by R. Carpenter, the authors have here produced a table of the modified Bessel functions of orders 0 and 1, for integer arguments ranging from 1 through 40. The tabular entries are presented in floating-point form and range in precision from 61S to 98S.

Standard power series for these Bessel functions were used in the underlying calculations, which were performed by multiple-precision arithmetic routines on an IBM 7094 system, the number of terms retained in the series ranging from 80 to 122, with increasing argument. The tabulated figures were subjected to the appropriate Wronskian check, which as the authors note, however, is satisfied even when erroneous values of Euler's constant and $\ln(x/2)$ have been used in the calculation of $K_n(x)$. Accordingly, the value of Euler's constant was carefully checked against several independent sources, and the computed natural logarithms were compared with those of Mansell [1].

The reviewer has compared the present basic tables with the tables of Aldis [2], and has thereby discovered in the latter, several errors which are listed elsewhere in this journal.

J. W. W.

1. W. E. Mansell, *Tables of Natural and Common Logarithms to* 110 *Decimals*, Royal Society Mathematical Tables, Vol. 8, Cambridge Univ. Press, New York, 1964. (See *Math. Comp.*, v. 19, 1965, p. 332, RMT **35.**)

2. W. S. Aldis, "Tables for the solution of the equation $d^2y/dx^2 + (1/x)\,dy/dx - (1 + n^2/x^2)y = 0$," *Proc. Roy. Soc. London*, v. 64, 1899, pp. 203–223.

35[7].—HENRY E. FETTIS & JAMES C. CASLIN, *A 20-D Table of Jacobi's Nome and its Inverse*, Report ARL 69-0050, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, March 1969, iv + 30 pp., 28 cm. [Copies obtainable from the Clearinghouse, U. S. Department of Commerce, Springfield, Va. 22151. Price $3.00.]

Jacobi's nome q is defined by the equation $q = \exp(-\pi K'/K)$, where K and K' are the quarter-periods of the Jacobian elliptic functions. The importance of this function derives from the fact that the Jacobian elliptic functions and theta functions possess well-known rapidly convergent expansions in terms of it.

This report consists of four tables: Table 1 gives q to 20D for $k^2 = 0.001(0.001)0.999$; Table 2 gives q to 20D for $\alpha = 0(0.1°)89°(0.01°)89.99°(0.0002°)90°$, where $\alpha = \sin^{-1} k$ is the so-called modular angle; Table 3 consists of 20D values of k and k' for $q = 0(0.001)0.5$; and Table 4 gives 20D values of q, $\bar{q}$, and $\bar{q}/q$ for $k' = 0.0001(0.0001)0.2$, as well as the second central differences of this tabulated ratio. The quantity $\bar{q}$ is defined as $\exp\{-\pi^2/[2\ln(4/k')]\}$; it approximates q with an error of the order of $(k')^2$, as the authors note in their explanatory remarks.

The tables were computed on an IBM 1620 system by means of modular reduction using Gauss's transformation, all arithmetical operations being carried to 23D prior to rounding the final results to 20D.

The user of these tables will probably be disconcerted by a series of derangements of tabular entries in Table 1 (p. 8), due to a corresponding disorder in the output cards used in the automatic printing.

Moreover, two of the five listed references contain errors. For example, the title of the important table of Curtis [1] is misquoted and the relevant paper of Salzer [2] is located erroneously in the *Journal*, instead of the *Communications*, of the *ACM*. The authors have informed this reviewer that they are planning to issue an appropriate errata sheet listing these corrections.

Despite such regrettable typographical imperfections, these tables constitute a significant improvement both in range and size of tabular interval over earlier tables of the Jacobi nome.

J. W. W.

1. A. R. CURTIS, *Tables of Jacobian Elliptic Functions whose Arguments are Rational Fractions of the Quarter Period*, National Physical Laboratory, *Mathematical Tables*, Vol. 7, Her Majesty's Stationery Office, London, 1964. (See *Math. Comp.*, v. 19, 1965, pp. 154–155, RMT **10**.)
2. H. E. SALZER, "Quick calculation of Jacobian elliptic functions," *Comm. ACM*, v. 5, 1962, p. 399.

36[7].—HENRY E. FETTIS & JAMES C. CASLIN, *Tables of Toroidal Harmonics*, I: *Orders* 0–5, *All Significant Degrees*, Report ARL 69-0025, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, February 1969, iv + 209 pp., 28 cm. [Copies obtainable from the Clearinghouse, U. S. Department of Commerce, Springfield, Va. 22151. Price $3.00.]

This report contains tables of both 11S and 16S values (all in floating-point form) of the toroidal harmonics, which are identifiable with the Legendre function of the second kind, $Q_\nu^\mu(s)$, when the order μ is an integer m, the degree ν is a half-odd integer $n - 1/2$, and the argument s exceeds unity.

The first table gives $Q_{n-1/2}^m(s)$ to 11S for $m = 0(1)5$, n varying from 0 through consecutive integers to a value (at least 29) for which the value of the function relative to that when n is zero is less than 10^{-12}, and $s = 1.1(0.1)10$.

The second table differs from the first with respect to the argument, which here is $\cosh \eta$, where $\eta = 0.1(0.1)3$. As noted in the abstract and explained in the introduction, this form of the argument appears naturally in the solution of the potential problem in toroidal coordinates.

The last two tables consist of 16S values of $Q_{n-1/2}^m(s)$ for $n = 0$ and 1, for the same range of values of m, s, and η as in the first two tables. These more extended decimal approximations were calculated independently by means of well-known formulas relating these toroidal functions to the complete elliptic integrals of the first and second kinds.

A useful introduction of 12 pages gives the derivation of these functions as solutions of Laplace's equation in toroidal coordinates, enumerates their principal properties, develops a continued fraction for the ratio of such functions of consecutive degree, and discusses the mathematical methods used in calculating the tables on IBM 1620 and IBM 7090 systems. Appended to the introduction is a list of five references.

Photographic offset printing of these tables from the computer sheets has not been completely satisfactory, as may be inferred from the two pages of corrigenda inserted to clarify a number of indistinctly printed tabular digits.

Despite such typographical imperfections, these extensive tables should prove generally useful to applied mathematicians.

J. W. W.

37[7].—M. Kumar & G. K. Dhawan, *Numerical Values of Certain Integrals Involving a Product of Two Bessel Functions*, Maulana Azad College of Technology, Bhopal, report and tables deposited in the UMT file.

In numerous applied problems, one encounters

$$I(\mu, \nu, \lambda) = \int_0^\infty e^{-pt} t^\lambda J_\mu(at) J_\nu(bt)\, dt.$$

A discussion of this integral with references to tables is given by Luke [1]. Let $a = t/h$; $b = u/h$; $u, t = 0.2(0.2)1.0$; $p = 2$; and $h = 1.05$, 1.10, 1.30 and 1.50. For all possible combinations of these parameters the authors tabulate $I(\mu, \nu, \lambda)$ to 6D for $\mu = \nu = 0, 1$, $\lambda = 1, 2, 3$, and for $\mu = 1$, $\nu = 0$ and $\lambda = 3$. All the integrals can be expressed in terms of the complete elliptic integrals of the first and second kinds. These expressions are delineated in an introduction to the tables.

Y. L. L.

1. Y. L. Luke, *Integrals of Bessel Functions*, McGraw-Hill Book Co., New York, 1962, pp. 314–318. (See also *Math. Comp.*, v. 17, 1963, pp. 318–320.)

38[9].—L. M. Chawla & S. A. Shad, "On a trio-set of partition functions and their tables", Table, *J. Natur. Sci. and Math.*, v. 9, 1969, pp. 87–96.

The main table here lists $q(n)$, $r(n)$, $\lambda(n)$, and $p(n)$ for $n = 1(1)150$. The last is the well-known partition function. The first is defined to be the number of partitions of n into *prime* parts. It is generated, of course, by

$$\sum_{n=0}^{\infty} q(n)x^n = [(1 - x^2)(1 - x^3)(1 - x^5)(1 - x^7)(1 - x^{11})\cdots]^{-1}.$$

Similarly, $r(n)$ is the number of partitions of n into *composites* and *unity*. Finally, $\lambda(n)$ is defined so as to take up the slack:

$$q(n) + r(n) + \lambda(n) = p(n).$$

A number of other notes in the same issue as this paper deal with these same functions and their generalizations.

The most interesting is $q(n)$, but this is not at all new. In [1] O. P. Gupta and S. Luthra give a longer table of this same function for $n = 1(1)300$. There is no reference to this earlier table here. The tables agree.

The obvious question is: How fast does $q(n)$ grow? One sees at once that $q(n)$ has a bit more than one-half the digits possessed by $p(n)$, and then that $q(n)/\sqrt{p(n)}$ appears to grow slowly with n. If one now examines $\log q(n)/\log p(n)$, one finds that this ratio is about $\frac{1}{2}$; it grows slowly, and *reaches a maximum* of 0.5572 at $n = 120$. Henceforth, the ratio very slowly decreases.

There is another function usually called $q(n)$, cf. [2]. Let us call it $Q(n)$ here. This is the number of partitions into *odd* parts. One knows theoretically that

$$\log Q(n)/\log p(n) \sim 1/\sqrt{2} = 0.7071.$$

As Morris Newman pointed out to me, this is consistent with the foregoing, since there are fewer primes than odd numbers, and therefore $Q(n)$ grows faster. As he also points out, the theory of $q(n)$ was given by Hardy and Ramanujan [3]. This gives

$$\log q(n)/\log p(n) \sim (2/\log n)^{1/2}$$

and explains the slow decrease that occurs after $n = 120$. In fact, after $n > e^8 \approx 3000$, $q(n)/\sqrt{p(n)}$ will no longer increase, but decrease slowly to 0.

D. S.

1. O. P. Gupta & S. Luthra, "Partition into primes," *Proc. Nat. Inst. Sci. India*, v. 21, 1955, pp. 181–184.
2. M. Abramowitz & I. A. Stegun, editors, *Handbook of Mathematical Functions*, Dover, New York, 1965; Section 24, "Combinatorial analysis" (see 24.2.1, 24.2.2, Table 24.5).
3. G. H. Hardy & S. Ramanujan, "Asymptotic formulae for the distribution of integers of various types," *Proc. London Math. Soc.*, (2), v. 16, 1917, pp. 112–132; see Eq. (5.281).

39[9].—Richard B. Lakein & Sigekatu Kuroda, *Tables of Class Numbers $h(-p)$ for Fields $Q(\sqrt{-p})$, $p \leqq 465071$*, University of Maryland, College Park, Md., November 1965, copy deposited in the UMT file.

The main table, which consists of 76 Xeroxed computer sheets, contains the class numbers $h(-p)$ for the first $19 \cdot 2^{10} = 19456$ primes of the form $4k + 3$, the largest of which is 465071. This table therefore goes much further than those of Ordman [1] and Newman [2], which have already been reviewed, although they were computed well after the present table.

The format is very unusual: the primes p and class number $h(-p)$ are listed on alternate pages. Every other page contains 512 primes in 16 columns and 32 rows, which are identified by numbers written in the base 32. One determines $h(-p)$ by using the same base 32 coordinates (on the next sheet) as those which identify p. Although the senior author had a rationale for such a curious format, we need not go into it; suffice it to say that it is usable.

We may now redo the lists in our previous two reviews and give definitive tables of the first and last p having $h(-p) = n$, $n = 1(2)49$, both for $p \equiv 7 \pmod 8$ and $p \equiv 3 \pmod 8$. It is very likely that the last examples listed are the largest that exist. But we cannot go further than $n = 49$ here, since several $h(-p) = 51$ exist with $p > 465071$. We also list the corresponding Dirichlet functions $L(1, \chi)$, cf. [3].

$\Delta = 8k + 7$

$h(-\Delta)$	first Δ	$L(1, \chi)$	last Δ	$L(1, \chi)$
1	7	1.18741	7	1.18741
3	23	1.96520	31	1.69274
5	47	2.29124	127	1.39386
7	71	2.60987	487	0.99651
9	199	2.00431	1423	0.74953
11	167	2.67414	1303	0.95735
13	191	2.95513	2143	0.88223
15	239	3.04819	2647	0.91593
17	383	2.72897	4447	0.80088
19	311	3.38472	5527	0.80289
21	431	3.17783	5647	0.87793
23	647	2.84070	6703	0.88256
25	479	3.58858	5503	1.05874
27	983	2.70543	11383	0.79503
29	887	3.05905	8863	0.96774
31	719	3.63201	13687	0.83245
33	839	3.57917	13183	0.90294
35	1031	3.42443	12007	1.00346
37	1487	3.01437	22807	0.76969
39	1439	3.22986	18127	0.91002
41	1151	3.79661	21487	0.87871
43	1847	3.14329	22303	0.90456
45	1319	3.89260	29863	0.81808
47	3023	2.68552	25303	0.92824
49	1511	3.96017	27127	0.93464

$\Delta = 8k + 3$

$h(-\Delta)$	first Δ	$L(1, \chi)$	last Δ	$L(1, \chi)$
1	3	0.60460	163	0.24607
3	59	1.22700	907	0.31294
5	131	1.37241	2683	0.30326

7	251	1.38807	5923	0.28574
9	419	1.38129	10627	0.27428
11	659	1.34617	15667	0.27609
13	1019	1.27940	20563	0.28481
15	971	1.51228	34483	0.25377
17	1091	1.61691	37123	0.27719
19	2099	1.30286	38707	0.30340
21	1931	1.50134	61483	0.26607
23	1811	1.69792	90787	0.23981
25	3851	1.26562	93307	0.25712
27	3299	1.47680	103387	0.26380
29	2939	1.68054	166147	0.22351
31	3251	1.70806	133387	0.26666
33	4091	1.62087	222643	0.21971
35	4259	1.68486	210907	0.23943
37	8147	1.28781	158923	0.29158
39	5099	1.71582	253507	0.24334
41	9467	1.32382	296587	0.23651
43	6299	1.70209	300787	0.24631
45	6971	1.69323	308323	0.25460
47	8291	1.62160	375523	0.24095
49	8819	1.63922	393187	0.24550

A second table deposited is listed on 9 pairs of sheets in the same format. This is a subtable, which includes only those p having

$$m^2 | h(-p) \qquad (m > 1).$$

It therefore includes all p having $h(-p) = 9, 25, 27, 45$, and 49, together with (incomplete) sets of p having $h(-p) = 63, 75$, etc. As indicated in our previous reviews, [1], [2], the desire to examine all 25's and 27's was the motivation for computing those tables. By examining the present table, I find, for example, that the two fields $Q(\sqrt{-p})$ have $h(-p) = 81$ for $p = 430411$ and 298483 (among many others). But these two have a class group $C(9) \times C(9)$, an elegant, but very unusual structure. I can now add these to $p = 134059$, that has the same group, which I found earlier.

D. S.

1. UMT **29,** *Math. Comp.*, v. 23, 1969, p. 458.
2. UMT **50,** *Math. Comp.*, v. 23, 1969, p. 683.
3. D. H. Lehmer et al., "Integer sequences having prescribed quadratic character," *Math. Comp.*, v. 24, 1970, pp. 433–451.

40[9].—Elvin J. Lee, *The Discovery of Amicable Numbers*, a 28-page history together with a computer-listed table of the 977 pairs of amicable numbers then known, Oak Ridge National Laboratory, Oak Ridge, Tenn., June 4, 1969, deposited in the UMT file.

Although this version is deposited in the Unpublished Mathematical Tables file, a revision will in fact be published, perhaps in several parts, in the *Journal of Recreational Mathematics.*

Since a revision will be published, we can be somewhat brief here. The text discusses all known methods of discovering amicable pairs, all known tables published, all the numerous errata and duplications therein, most of the theory developed, most of the unsolved problems announced, and it concludes with a 61-item bibliography.

Of the 977 pairs, the oldest (220 and 284) has an unknown discoverer, but each of the remaining pairs is attributed to its (apparent) discoverer. The grand totals for each discoverer are: Fermat, 1; Descartes, 1; Euler, 59; Legendre, 1; Paganini, 1; Seelhoff, 2; Dickson, 2; Mason, 14; Poulet, 104; Gerardin, 5; Poulet & Gerardin, 4; Escott, 218; Brown, 1; Garcia, 150; Rolf, 1; Alanen, Ore & Stemple, 8; Lee (himself), 390; Bratley & McKay, 14.

In the revision, it appears that Garcia gets three more, and 59 new pairs (at least) will be credited to Cohen, and, almost simultaneously, to Bratley, McKay and Lunnon.

Besides these attributions, each pair in the table has an identifying number, and the ratio of its two members is also listed. The 977 pairs are listed by type, not magnitude. The first type is

$$E\,p, \qquad E\,qr$$

which means that the common factor E is prime to p, q, and r, and these latter are distinct primes. The next type is

$$E\,pq, \qquad E\,rs,$$

etc. Within types the ordering is, first, by the power of 2 in E, and then by some similar rules. However, the revision will reorder some of these pairs, since in this edition there is no explicit set of rules that unequivocally defines the location of each pair.

While such a listing (by types) is more analytical than a numerical listing in that it is more amenable to theory, it does have the drawback that new discoveries must be interpolated, thereby destroying the identifying numbers.

While the text and table will be revised, owing mainly to the new surge of work in this field, it certainly is a must for everyone interested in this field, and Lee has a few extra copies.

One observation (conjecture?) here has already bitten the dust because of this new work. The operator

$$\sigma_0(N) = \sigma(N) - N$$

defines perfect numbers by $\sigma_0(N) = N$, amicable numbers by $\sigma_0^2(N) = N$, and sociable numbers of order k by $\sigma_0^k(N) = N$. Poulet long ago found one example each of $k = 5$ and $k = 28$, and until recently no other $k > 2$ were known. Now it happens that there is no solution of $\sigma_0(x) = 2$ or $\sigma_0(x) = 5$ or $\sigma_0(x) = 28$, and it was thought that this fact was somehow associated with the known existing k's: 2, 5, 28. On the other hand, one has $\sigma_0(16) = \sigma_0(33) = 15$, $\sigma_0(15) = 9$, $\sigma_0(9) = 4$, and $\sigma_0(4) = 3$, and no sociables of order $k = 4$ or 3 were known. That was the observed pattern. But Cohen then found nine examples of $k = 4$.

D. S.

41[9].—D. H. LEHMER, *Tables of Ramanujan's Function* $\tau(n)$, 1963, ms. of 164 pages of computer printout deposited in the UMT file.

There are listed here the values of Ramanujan's $\tau(n)$ for $n = 1(1)10^4$. This is defined [1] by

$$(1) \qquad \sum_{n=0}^{\infty} \tau(n)x^n = x\left\{\prod_{n=1}^{\infty}(1 - x^n)\right\}^{24}$$

and begins: $\tau(1) = 1$, $\tau(2) = -24$, $\tau(3) = 252, \cdots$. It was computed by the known [1] recurrence formula obtained by logarithmically differentiating (1). This recurrence calculation was run (about 1963) on Stanford's IBM 7094, and values were checked by the relation

$$\tau(mn) = \tau(n)\tau(m) \qquad \text{for } (m, n) = 1.$$

Earlier tables of $\tau(n)$ were by Watson [2] to 1000, Lehmer [3] to 2500, and Ashworth & Atkin [4] to 1921.

Watson also listed

$$\tau^*(n) = \tau(n)n^{-11/2}$$

to test Ramanujan's conjecture [1]:

$$|\tau^*(n)| < d(n),$$

where $d(n)$ is the number of divisors of n. For primes p, this becomes

$$|\tau^*(p)| < 2,$$

and Watson found

$$\tau^*(103) = -1.91881, \qquad \tau^*(479) = +1.90410.$$

Lehmer does not list $\tau^*(n)$ here, but casting an eye along his (long) column of numbers I found

$$\tau^*(3371) = +1.94978, \qquad \tau^*(3967) = +1.95144,$$

$$\tau^*(7451) = -1.95502, \qquad \tau^*(7589) = +1.96053.$$

The apparent existence of sequences p with

$$\tau^*(p) \to +2 \quad \text{and} \quad \tau^*(p) \to -2,$$

from below, and above, respectively, does suggest the stronger conjecture that $+2$ and -2 are the *best possible* bounds. The function

$$\cos\theta_p = \tfrac{1}{2}p^{-11/2}\tau(p)$$

enters the theory, (see [5] for an introductory account), and we record

$$\theta_{7589} = 11°24'$$

for the smallest angle noted.

Also of interest would be values of

$$|\tau^*(p)| \approx 1,$$

or

$$\theta_p \approx \pi/3, 2\pi/3,$$

since

$$\tau(p^2) = (\tau(p))^2 - p^{11}$$

and therefore $\tau(p^2)$ is relatively very small. An early example is

$$\tau^*(11) = 1.0087, \qquad \tau^*(121) = 0.00175,$$

but these values are more difficult to pick out merely by glancing down the column of $\tau(n)$.

Subsequently, the original table deposited was replaced by a second that also lists the summatory function $\sum^N \tau(n)$.

D. S.

1. S. RAMANUJAN, "On certain arithmetical functions," *Trans. Cambridge Philos. Soc.*, v. 22, 1916, pp. 159–184; see especially §§16–18. A short table of $\tau(n)$ for $n = 1(1)30$ is given here.
2. G. N. WATSON, "A table of Ramanujan's function $\tau(n)$," *Proc. London Math. Soc.*, (2), v. 51, 1950 (paper is dated 1942), pp. 1–13.
3. D. H. LEHMER, *Tables of Ramanujan's* $\tau(n)$, UMT **101**, *MTAC*, v. 4, 1950, p. 162.
4. MARGARET ASHWORTH & A. O. L. ATKIN, *Tables of* $p_k(n)$, UMT **1**, *Math. Comp.*, v. 21, 1967, p. 116.
5. G. H. HARDY, *Ramanujan*, Chelsea reprint, New York, 1959, Chapter X and §§9.17, 9.18.

42[9].—PAUL TURÁN, Editor, *Number Theory and Analysis—A Collection of Papers in Honor of Edmund Landau* (1877–1938), Plenum Press, New York, 1969, 355 pp., 24 cm. Price $19.50.

There are 22 papers here in number theory and analysis in honor of Landau by E. Bombieri, H. Davenport, B. M. Bredihin, J. V. Linnik, N. G. Tschudakoff, J. G. van der Corput, M. Deuring, P. Erdös, A. Sárközi, E. Szemerédi, H. Heilbronn, E. Hlawka, A. E. Ingham, V. Jarnik, S. Knapowski, P. Turán, J. Kubilius, J. E. Littlewood, L. J. Mordell, G. Pólya, J. Popken, H. Rademacher, A. Rényi, I. J. Schoenberg, C. L. Siegel, Arnold Walfisz, and Anna Walfisz.

There also is a joint paper by Davenport and Landau himself: "On the representation of positive integers as sums of three cubes of positive rational numbers". Davenport explains: "This paper was written, in a rough form, in February 1935, when Landau visited Cambridge . . . As far as I can recollect, the reason why the paper was not published. . . ." Unfortunately, Landau is not the only departed author here, since one must add Ingham, Knapowski, Arnold Walfisz, Rademacher, and Davenport.

The papers are in English and German. The volume first appeared in Germany with the title "Abhandlungen aus Zahlentheorie und Analysis zur Erinnerung an Edmund Landau (1877–1938)". It includes a photograph of Landau, a short foreword by Turán, and a list of Landau's seven books and his 254 papers (not counting the one here).

Landau's first two papers (published at age 18) were on chess, but with the third he begins his real life work. It is his well-known 1899 Inaugural Dissertation: "Neuer Beweis der Gleichung $\sum_1^\infty \mu(k)/k = 0$". Landau liked to joke about this paper. "Gordan pflegte etwa zu sagen: 'Die Zahlentheorie ist nützlich, weil man nämlich mit ihr promovieren kann'. Ich habe mit einer Antwort auf diese Frage 1899 promoviert."

The lead-off paper in the book is by Bombieri and Davenport, "On the large sieve method". Most of the other papers are on number theory, but (about) $1/\pi$ of them are on analysis, more exactly, 7 out of 22 of them.

D. S.

43[10]—ROBERT SPIRA, *Cyclotomic Polynomial Generator and Tables*, Version A, Michigan State University, East Lansing, Mich., October 1969, 45 pp., 28 cm., deposited in UMT file.

This is an emended version of an undated report released several months earlier, which was found to contain several serious typographical errors in the arrangement of the tabular entries.

The numerical table herein consists of a parallel listing of values of the Euler totient, $\phi(n)$ and of the coefficients of the cyclotomic polynomial $Q_n(x)$ for $n = 1(1)250$. (This polynomial is defined as the irreducible monic polynomial of degree $\phi(n)$ that has as its zeros the primitive nth roots of unity.)

This main table is preceded by a detailed description (including flow charts and listings) of the four FORTRAN programs used in the calculations.

The introduction describes the mathematical procedure followed in the generation of the cyclotomic polynomials, which the author ascribes to Lehmer [1].

J. W. W.

1. D. H. LEHMER, *Guide to Tables in the Theory of Numbers*, Bulletin No. 105, National Research Council, Washington, D. C., 1941, p. 73.

44[12]—J. HILSENRATH, G. G. ZIEGLER, C. G. MESSINA, P. J. WALSH & R. J. HERBOLD, *Omnitab, A Computer Program for Statistical and Numerical Analysis*, National Bureau of Standards Handbook No. 101, 1966, reissued 1968 with corrections, lx + 275 pp., 26 cm. Price $3.00.

Computing has come a long way from the early beliefs of von Neumann that a computer user will be a scientist who will know the range of every number entering in his calculation, and who will be so motivated that machine language programming will present no problem. In fact, even the use of floating-point arithmetic was considered to be "playing with fire". Today we find a veritable Tower of Babel of languages, collections of algorithms, subroutine libraries and operating systems. and the promise of a console in every home for doing Junior's homework and to facilitate menu preparation. It is therefore hard to realize that there exist large numbers of problem-oriented research people who want access to a large digital computer, but who do not want to learn programming, for example, they may just want "a least-squares fit". For these people large numbers of packages and "general-purpose" systems have been devised.

OMNITAB, developed by the Statistical Engineering Laboratory of the National Bureau of Standards, is a completely assembled interpretive program which provides facilities for doing a wide range of statistical and engineering type calculations. Originally written for the IBM 7090/7094, this volume is a manual for users with access to a computer with the OMNITAB system and indicates in detail the necessary control cards, rules for carrying out computations and gives numerous examples. Instructions are given to the system in a form of English sentences simulating desk computing. Indicative of the popularity of this system, is the fact that subsequent to the issuance of this volume, the system has been rewritten in ASA Fortran and implemented in several other computers such as the Univac 1108, IBM 360-50 and 65 and CDC 6400 and 6600. While the volume under review indicates control cards for the 7094 only, various versions are in use throughout the country and a version has even been unveiled recently in a time-sharing environment.

As an old-fashioned "programming expert", this reviewer has a certain antipathy toward the concept of "instant programs"—I probably feel that one should have enough motivation to write the appropriate subroutine, if one wants to use anything beyond the arithmetic operators and elementary functions. However, the audience is there, and systems such as OMNITAB have served a real need. The user of this or similar systems will be well advised to read the section entitled diagnostic features. What is pointed out there, cannot be repeated too often:

"The concept of a general-purpose program rests in some measure on the assumption that the user, though not a programmer, is familiar with the behavior of the mathematical functions he is using or trying to compute . . . diagnostic features are incorporated . . . however diagnostic statements are no substitute for sound mathematical analysis, which is necessary to avoid the more serious pitfalls of numerical computations."

In this regard, it would have been useful to list the possible diagnostic statements generated by each command. In fact, the numerical methods used in the functions and routines should have been given. Not only would they be useful to users outside of OMNITAB, but they are a must when trying to gauge numerical accuracy. In the final analysis, it probably does not matter whether one learns to program or to use "packages" or both. Of paramount importance is the question of the accuracy of the results. The hope is that all users of computers in future generations, those in the physical as well as those in the social sciences, will learn the elements of numerical analysis.

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45[12].—F. R. A. HOPGOOD, *Compiling Techniques*, American Elsevier Publishing Co., Inc., New York, 1969, vi + 123 pp., 23 cm. Price $6.00.

This elegant monograph, suitable both for self-study and for text use in short courses on compiling, manages in its brief (126 pages) compass to discuss many of the most salient issues of compiler writing in an illuminating manner. An introductory section (28 pages) discusses compiler-related data structures and their computer treatment, and includes a thumbnail account of hashing. Backus notation is then introduced. Lexical analysis is discussed in a 10-page chapter which is, unfortunately, less transparent than other passages of the book. Many of the principal parsing methods are then nicely surveyed in a 20-page chapter. Code generation is next discussed, with emphasis on arithmetic expression code generation. This chapter includes a discussion of fundamental code-block optimizations, including redundant store elimination and common subexpression finding, with a following chapter discussing straight-line register allocation and temporary storage minimization in more detail. The book ends with a quick comparative survey of various compiler-writing systems.

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46[12].—P. J. KIVIAT, R. VILLANEUVA & H. M. MARKOWITZ, *The SIMSCRIPT* II *Programming Language*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1969, xiii + 386 pp., 25 cm. Price $10.95 cloth, $6.95 paper.

SIMSCRIPT is a programming language whose primary orientation is towards the programming of computer simulations, but which has the facilities of a general-purpose language. The authors of this book have chosen to emphasize the general-purpose aspects of SIMSCRIPT rather than its simulation capabilities.

Stylistically, SIMSCRIPT is a very "smooth" language, and its design is highly professional. The syntax, like that of COBOL, is intended to make programs read like English sentences, though briefer modes of expression are permitted. With the syntax stripped away, the algebraic part of the language would look like FORTRAN with a few ALGOL features, such as conditional statements and DO loops with variable parameters. There are additional facilities for text manipulation and some rather elaborate report-generating capabilities (quite useful, of course, in simulation experiments). The input-output is well planned and easy to use.

It is doubtful that SIMSCRIPT would attract many users, however, solely on the basis of its general-purpose facilities. The strength of the language lies in its capabilities for handling entities, sets and attributes. An *entity* is a computational object capable of having attributes, of belonging to sets, and of owning sets; an owned set may be thought of as a set-valued attribute. The attribute facility may be used to create PL/I-like structures, though the set operations have no immediate PL/I counterpart. Entities may be removed from or added to sets in a number of different ways, corresponding to various forms of queueing.

The simulation facilities are based upon the notions of a system clock, which keeps track of simulated time, and of events which are computations to be carried out at a certain point in simulated time. Events may arise either endogonously (internally generated) or exogonously (externally generated). After all events associated with the current time are executed, the system clock is simply advanced to the time of the next scheduled event or events. Execution of an event, of course, can cause the creation of new events, which may be scheduled at the current time or at later times.

I disagree with the authors' claim that the general-purpose part of SIMSCRIPT is comparable in power with ALGOL or PL/I. For instance, SIMSCRIPT does not have the ALGOL block structure, though it does permit recursive functions. It also lacks certain conveniences such as the ability to start array subscripts at values other than one. The character string variables are of two types: ALPHA variables, which fit into a single computer word, and TEXT variables, which are arbitrarily long strings. The distinction between the two is thus implementation-based, and could be confusing to the neophyte.

The implementation of SIMSCRIPT uses dynamic storage allocation for entities, and for arrays as well. The result has a great deal of flexibility, though I suspect some of it at the price of efficiency. The SIMSCRIPT "DO" statement, like the ALGOL "FOR" statement, permits the parameters of the "DO" to vary as the loop is executed; thus, unless the compiler is very clever about it, execution of "DO" loops will involve a great deal of unnecessary recomputation.

The book is written in five sections, in order of increasing difficulty. They are described by the authors as:

1. A simple teaching language designed for nonprogrammers.
2. A language comparable in power to FORTRAN.
3. A language comparable in power to ALGOL or PL/I.
4. The entity-attribute-set features of SIMSCRIPT.
5. The simulation-oriented part of SIMSCRIPT.

This arrangement was made for pedagogical reasons, but it does not quite succeed. An experienced programmer will find the book slow reading if he tries to master all the details; a novice would have a great deal of difficulty. However, the information is all there, and a determined reader will assimilate it sooner or later.

I would recommend this book for those interested in the field of programming languages, for those who need to write a computer simulation and are shopping around for a language in which to write it, and for those who are working on problems where the set and property manipulation facilities of SIMSCRIPT would be helpful.

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47[12].—JEAN E. SAMMET, *Programming Languages: History and Fundamentals*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1969, xxx + 785 pp., 24 cm. Price $18.00 ($13.50 student edition).

Programming Languages: History and Fundamentals is a monumental and encyclopedic treatment of its field. The primary aim of the book, as stated by the author, is to provide, ". . . in one place, and in a consistent fashion, fundamental information on programming languages, including history, general characteristics, similarities, and differences." This aim has without doubt been achieved, and achieved well. A second aim is ". . . to provide specific basic information on all the significant, and most of the minor, higher level languages developed in the United States." This aim also has been achieved; there is an impressive collection of about 120 languages described in varying amounts of detail.

The book is organized into three introductory chapters, six chapters on languages grouped more or less according to application areas, and two chapters on unimplemented concepts and future long-range developments. The organization is well thought out and suits the author's purposes admirably. The first chapter is devoted to defining what is meant by a programming language, to pros and cons of higher-level languages, to classifying and categorizing languages, and to a presentation of factors influencing the choice of a language. Omitted from consideration are most European programming languages, and certain languages that are not, by the author's definition, higher-level programming languages, e.g., Report Generator, Autocoder, APL and SLIP. The second and third chapters are devoted to describing functional and technical characteristics of programming languages, partly with the aim of developing a format for the later discussions of specific languages. Functional characteristics, roughly, are the historical, political, economic, and pragmatic aspects of a language, i.e., those that are not part of its definition. Technical characteristics have to do with the actual syntax and semantics of the language. This outline of discussion is spelled out in some detail and generally adhered to in the sequel. The fact that approximately equal amounts of space are devoted to functional and technical characteristics is an indicator of the tone of the discussion. The author does not attempt to teach the reader how to program in any of the languages discussed, and indeed, one certainly would not learn programming from *this* book.

As a reference work for background on specific programming languages, the book is a success. As a textbook on programming languages as a field of study, it is not. The tone of the introductory chapters is too platitudinous, and the technical information, both in depth and in organization, is inadequate to the task of imparting a feeling for what programming languages are all about. The difficulty is that this feeling can ultimately be achieved only by actually writing programs in various languages and thus coming to appreciate their fine points. Thus, the uniformity of organization across languages is achieved at the cost of omitting really deep examination of a few of them. The chapter on technical characteristics attempts to raise the issues of what choices a language designer must make, but the question of how, historically, they have been made is scattered among the different language discussions and never really explored in depth. Questions such as scoping rules, extensibility, and the structuring of data aggregates are never treated in a unified way, and the material on minor languages and on functional characteristics tends to de-emphasize the importance of the technical characteristics of the major languages. There are no exercises in the book, and, indeed, the material provides no basis for exercises.

In summary: *Programming Languages: History and Fundamentals* is a fine source for factual knowledge, but a poor source for gaining understanding.

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48[12].—M. V. WILKES, *Time-Sharing Computer Systems*, American Elsevier Publishing Co., Inc., New York, 1968, iv + 102 pp., 22 cm. Price $4.95.

This is an excellent introduction to the design of time-sharing systems, which expresses concisely a number of current ideas. The text is clearly influenced by the pioneering work done at Massachusetts Institute of Technology and the time-sharing system implemented on the Atlas at Cambridge University. A number of addressing schemes, including the paging and segmentation features provided on the GE 645 and IBM 360/67, are discussed, as well as the interesting work, on the generalized notion of a capability, of Dennis and Van Horn, and Yngve and Fabry.

The serious student will undoubtedly wish to refer to the references cited for more details.

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