

A New Method for Computing Toroidal Harmonics

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Abstract. A method for computing Legendre functions of integer order and half-odd degree is presented. The method is based on the theory of quadratic transformation of the argument, and is a generalization of Gauss' or Landen's transformation for computing elliptic integrals.

Legendre functions of integral order and half-odd degree are sometimes referred to as "Toroidal Harmonics." The latter name originates from the fact that these functions arise in the solution of Laplace's equation in toroidal coordinates ξ, η, ϕ . Separation of variables in this coordinate system leads, for the determination of one of the components, to the differential equation

$$(1) \quad (1 - s^2) \frac{d^2 y}{ds^2} - 2s \frac{dy}{ds} + \left[\left(\nu^2 - \frac{1}{4} \right) - \frac{\mu^2}{1 - s^2} \right] y = 0 \quad (s = \cosh \xi),$$

while the remaining components have simple expressions of the form $e^{i\nu\eta}, e^{i\mu\phi}$, where μ and ν must be integers in order to obtain single-valued solutions. The solution of Eq. (1) can be given in terms of Legendre functions:

$$(2) \quad y = AP_{\nu-1/2}^{\mu}(s) + BQ_{\nu-1/2}^{\mu}(s).$$

The functions $P_{\nu-1/2}^{\mu}(s)$ have the property that they are finite as $s \rightarrow 1$ but increase with increasing ν , while the functions $Q_{\nu-1/2}^{\mu}(s)$ are singular for $s \rightarrow 1$ but tend to zero as $\nu \rightarrow \infty$. Thus, in an exterior problem, it is customary to use the $Q_{\nu-1/2}^{\mu}$ with fixed μ and increasing ν .

For $\nu = 0$ and 1, $Q_{\nu-1/2}^{\mu}(s)$ can be expressed in terms of complete elliptic integrals, and hence higher degrees could, in theory, be computed by the known recurrence relation for these functions. However, as is well known, this recurrence relation is unstable, and poor or even meaningless values will be obtained for the higher degrees. This phenomenon also occurs in computing Bessel functions of higher order, and, as an alternative, J. C. P. Miller devised a scheme whereby the recurrence relation was employed in the backward direction (in which case no loss of accuracy is incurred), starting with values of the higher orders determined by some other means. Miller's algorithm was proposed by Rotenberg [1] as a means of computing toroidal harmonics, and used by Wright and Peterson [2] in preparing a brief table of these functions. The general theory behind this process has been the subject recently of some more detailed study by Gautschi [3] and Wimp [4]. A modification of the method used by the present author [5] employs a continued fraction to obtain the ratio of higher-degree terms.

The method described above has the disadvantage that all of the significant higher-degree functions must be computed first even if only a few of lower degree

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are needed. For s near 1, the convergence is very slow. As an example, when $s = 1.01$, over 100 terms of the continued fraction and functions up to degree 200 are required to be assured of 10-figure accuracy.

The present method is one which permits the functions to be computed directly starting with those of lowest degree and continuing as far as is desired. It is based on the theory of quadratic transformation, and is entirely analogous to the well-known Gauss or Landen transformation for elliptic integrals. Functions with arguments s are expressed in terms of those with argument s_1 where $s_1 > s$, and the transformation is repeated until an argument is obtained which is sufficiently large to allow the functions to be expressed by elementary functions. From these last results, those with the original argument can be calculated. The procedure can be applied with relatively no loss in accuracy.

The required transformation can be obtained from the two representations ([7, formula 3.671(2)], [8, Chapter IV, Section 5]).

$$(3) \quad Q_{n-1/2}(s) = 2k^{n+1/2} \int_0^{\pi/2} \frac{\cos^{2n} \theta \, d\theta}{(1 - k^2 \cos^2 \theta)^{1/2}},$$

$$(4) \quad Q_{n-1/2}(s_1) = k^{-1} \int_0^{\pi/2} \frac{\cos 2n\theta \, d\theta}{(1 - k^2 \cos^2 \theta)^{1/2}},$$

where

$$(5) \quad s = (1 + k^2)/2k$$

and

$$s_1 = \frac{2}{k^2} - 1.$$

By using the relationship

$$(6) \quad 2^{2n-1} \cos^{2n} \theta = \left[\frac{1}{2} \binom{2n}{n} + \sum_{i=1}^n \binom{2n}{n-i} \cos 2i\theta \right],$$

we obtain

$$(7) \quad Q_{n-1/2}(s) = \frac{[s + (s^2 - 1)^{1/2}]^{1/2-n}}{2^{2n-1}} \left[\binom{2n}{n} Q_{-1/2}(s_1) + 2 \sum_{i=1}^n \binom{2n}{n-i} Q_{i-1/2}(s_1) \right],$$

with

$$(8) \quad s_1 = [s + (s^2 - 1)^{1/2}][s + 3(s^2 - 1)^{1/2}].$$

For $n = 0$ and 1 the above gives

$$(9) \quad \begin{aligned} Q_{-1/2}(s) &= 2[s + (s^2 - 1)^{1/2}]^{1/2} Q_{-1/2}(s_1), \\ Q_{1/2}(s) &= \frac{1}{[s + (s^2 - 1)^{1/2}]^{1/2}} [Q_{-1/2}(s_1) + Q_{1/2}(s_1)] \end{aligned}$$

which, as can be easily verified, are equivalent to Gauss' transformation for elliptic integrals, upon making use of the relations

$$(10) \quad \begin{aligned} Q_{-1/2}(s) &= 2[s + (s^2 - 1)^{1/2}]^{-1/2} K[s - (s^2 - 1)^{1/2}], \\ Q_{1/2}(s) &= 2[s + (s^2 - 1)^{1/2}]^{1/2} \{ K[s - (s^2 - 1)^{1/2}] - E[s - (s^2 - 1)^{1/2}] \}. \end{aligned}$$

It is clear that $s_1 > s$ and that, by repeating the transformation, s_m will ultimately be sufficiently large for the function $Q_{\nu-1/2}(s_m)$ to be approximated from the formula

$$(11) \quad Q_{n-1/2}(s_m) \cong \frac{\sqrt{\pi} \Gamma(n + \frac{1}{2})}{\Gamma(n + 1)} [s_m + (s_m^2 - 1)^{1/2}]^{-(n+1/2)},$$

which is accurate to within a relative error of the order of $(1/2s_m)^2$.

Having found $Q_{n-1/2}(s_m)$, Eq. (7) may then be used to find $Q_{n-1/2}(s_{m-1})$, continuing until the required values of $Q_{n-1/2}(s)$ are obtained.

Finally functions of higher order can be obtained recursively from those of zero order by means of the formula

$$(12) \quad (s^2 - 1)^{1/2} Q_{\nu+1/2}^{\mu+1}(s) = (\nu - \mu + \frac{1}{2})s Q_{\nu+1/2}^{\mu}(s) - (\nu + \mu + \frac{1}{2})Q_{\nu-1/2}^{\mu}(s),$$

for $\nu = 0, 1, 2, \dots$ and from

$$(s^2 - 1)^{1/2} Q_{-1/2}^{\mu+1}(s) = (\frac{1}{2} - \mu)Q_{1/2}^{\mu}(s) - (\frac{1}{2} + \mu)s Q_{-1/2}^{\mu}(s),$$

for $\nu = -1$.

Numerical Example. To calculate $Q_{n-1/2}(s)$ for $s = 1.01$ and $n = 0, 1, 2, 3, 4, 5$. We start by computing s_i according to Eq. (8); until a value is obtained which is sufficiently large for Eq. (11) to apply:

i	s_i		
0	1.01		
1	1.65316	88539	02
2	1.66369	41381	96×10
3	2.20930	07386	65×10^3
4	3.90480	73030	93×10^7
5	1.21980	16059	43×10^{16}

The values of $Q_{n-1/2}(s_5)$ as given by Eq. (11) will now be accurate to better than 30 figures:

n	$Q_{n-1/2}(s_5)$		
0	2.01136	21555	09×10^{-8}
1	4.12231	41240	83×10^{-26}
2		—	
3		—	
4		—	
5		—	

(Note that functions for $n \geq 2$ are of no significance, since their magnitude relative to $Q_{-1/2}$ is negligible.)

Equation (7) is now used to compute $Q_{n-1/2}(s_i)$ for the remaining values of the s_i :

n	$Q_{n-1/2}(s_4)$			$Q_{n-1/2}(s_3)$		
0	3.55496	28891	00×10^{-4}	4.72614	96885	28×10^{-2}
1	2.27601	68512	56×10^{-12}	5.34801	56392	27×10^{-6}
2	2.18578	34535	01×10^{-20}	9.07755	98659	40×10^{-10}
3		—		1.71199	72588	50×10^{-13}
4		—		3.39020	77090	78×10^{-17}
5		—		6.90532	31240	82×10^{-21}

n	$Q_{n-1/2}(s_2)$			$Q_{n-1/2}(s_1)$		
0	5.44995	36858	06×10^{-1}	1.87832	66753	09
1	8.19787	45943	47×10^{-3}	3.21017	38150	62×10^{-1}
2	1.84955	91684	76×10^{-4}	8.14856	90453	27×10^{-2}
3	4.63643	83364	18×10^{-6}	2.29243	39889	94×10^{-2}
4	1.22035	66435	80×10^{-7}	6.76530	09967	33×10^{-3}
5	3.30386	17771	74×10^{-9}	2.05248	65657	25×10^{-3}

Finally we get:

n	$Q_{n-1/2}(s)$		
0	4.03166	87796	89
1	2.04931	84514	62
2	1.41585	92547	73
3	1.05843	74848	35
4	8.21280	86319	17×10^{-1}
5	6.51426	26170	33×10^{-1}

As noted earlier, over 200 terms would have been required to obtain comparable accuracy by either of the two methods described earlier in the paper.

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1. A. ROTENBERG, "Calculation of toroidal harmonics," *Math. Comp.*, v. 14, 1960, pp. 274-276. MR 22 #6066.
2. E. B. WRIGHT & P. I. PETERSON, *Brief Table of Ring Functions*, NRL Memorandum Report 1969, U. S. Naval Research Laboratory, March 1966.
3. W. GAUTSCHI, "Computational aspects of three-term recurrence relations," *SIAM Rev.*, v. 9, 1967, pp. 24-82. MR 35 #3927.
4. J. WIMP, *Recent Developments in Recursive Computation*, ARL 69-0104, Aerospace Research Laboratories, July 1969.
5. HENRY E. FETTIS, *Calculation of Toroidal Harmonics without Recourse to Elliptic Integrals*, Blanch Anniversary Volume, Aerospace Research Lab., U. S. Air Force, Washington, D.C., 1967, pp. 21-34. MR 35 #2460.
6. HENRY E. FETTIS & J. C. CASLIN, *Tables of Toroidal Harmonics*, ARL 69-0025, Aerospace Research Laboratories, February 1969. MR 39 #6481.
7. I. S. GRADSHTEYN & I. M. RYZHIK, *Table of Integrals, Series, and Products*, Fizmatgiz, Moscow, 1963; English transl., Academic Press, New York, 1965. MR 28 #5198; MR 33 #5952.
8. W. MAGNUS & F. OBERHETTINGER, *Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik*, 2nd ed., Springer, Berlin, 1948; English transl., Chelsea, New York, 1949. MR 10, 38; MR 10, 532.