

The Explicit Inverse of a Tridiagonal Matrix

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Abstract. The closed form inverse of a tridiagonal matrix, which is a slight generalization of a matrix considered by D. Kershaw (*Math. Comp.*, v. 23, 1969, pp. 189-191), is given in this note. If the matrix has integer elements, an integer multiple of the inverse can be computed by integer arithmetic, that is, without machine roundoff error.

Forms of B and B⁻¹. Let $B = (b_{ij})$ be an $n \times n$ matrix given by

$$(1) \quad \begin{aligned} b_{ij} &= b_i, & i &= j, \\ &= \delta_{i,i-1}, & i &< j, \\ &= \delta_{i-1,j}, & i &> j, \end{aligned}$$

where $b_i = b_{n-i+1}$ and δ_{ij} is the Kronecker delta. Define $r_k = -(b_k r_{k-1} + r_{k-2})$, $k = 2, \dots, n-1$, and $r = (b_n r_{n-1} + r_{n-2})$, where $r_0 = 1$ and $r_1 = -b_1$. Let $C = (c_{ij})$ be an $n \times n$ matrix, where

$$(2) \quad \begin{aligned} c_{ij} &= r^{-1} r_{i-1} r_{n-j}, & i &\leq j, \\ &= c_{ji}, & i &> j, \end{aligned}$$

then $C = B^{-1}$. An elementary but lengthy proof of this is to show that BC is an $n \times n$ identity matrix. It can also be shown that $r = (-1)^{n+1} \det(B)$. This implies that $r_{i-1} r_{n-j} = (-1)^{n+1} A_{ji}$, where A_{ji} is the cofactor of b_{ji} .

If $b_i, i = 1, \dots, n$, is an integer, then $r_k, k = 2, \dots, n-1$, and r are integers. This implies that $rc_{ij} = r_{i-1} r_{n-j}$ is an integer, that is, an integer multiple of B^{-1} can be generated by integer arithmetic. Similarly, if one had a linear system to solve, which was represented by a matrix of the form (1) with integer elements, then an integer multiple of the solution could be computed by integer arithmetic if the known vector had rational coordinates.

Generalization to Partitioned Matrices. If the elements $b_i, i = 1, \dots, n, 1, 0$ of B are replaced by $m \times m$ matrices $B_i, i = 1, \dots, n, I, 0$, where I is the identity matrix and 0 is the null matrix, respectively, then the inverse of B is given by the partitioned matrix (2) if $B_i B_j = B_j B_i, i, j = 1, \dots, n$. Here r^{-1} is considered the inverse of some $m \times m$ matrix.

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