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Chebyshev Type Quadrature Formulas*

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Abstract. Quadrature formulas of the form

$$\int_{-1}^{1} f(x)\,dx \approx \frac{2}{n}\sum_{i=1}^{n} f(x_i^{(n)})$$

are associated with the name of Chebyshev. Various constraints may be posed on the formula to determine the nodes $x_i^{(n)}$. Classically the formula is required to integrate nth degree polynomials exactly. For $n = 8$ and $n \geqq 10$ this leads to some complex nodes. In this note we point out a simple way of determining the nodes so that the formula is exact for polynomials of degree less than n. For $n = 8$, 10 and 11 we compare our results with others obtained by minimizing the l^2-norm of the deviations of the first $n + 1$ monomials from their moments and point out an error in one of these latter calculations.

In a recent paper Barnhill, Dennis and Nielson [1] considered the possibility of finding quadrature formulas of the form

$$\int_{-1}^{1} f(x)\,dx \approx \frac{2}{n}\sum_{k=1}^{n} f(x_k) \tag{1}$$

with x_k symmetric in $[-1, 1]$ so that

$$\sigma_n = \sum_{j=0}^{n}\left[\frac{2}{n}\sum_{k=1}^{n} x_k^j - m_j\right]^2$$

is minimized, where $m_j = \int_{-1}^{1} x^j\,dx$. They have computed solutions numerically for $n = 8$, 10 and 11. Classically the x_i had been determined so (1) is exact for $1, x, \cdots, x^n$ but this leads to some complex nodes if $n = 8$ or $n \geqq 10$.

Another possibility is to consider formulas (1) with the x_i chosen so (1) is exact for $1, x, \cdots, x^p, p < n$ with $x_i \in [-1, 1]$. This problem does not have a solution if n and p are both required to be large [2], [3]. Nevertheless, for the small n considered here, this problem has easily computed solutions. Although σ_n will not in general be minimized, the resulting formulas have a certain appeal since polynomials of degree p or less can be integrated exactly.

If (1) is to be exact for $1, x, \cdots, x^p, p < n$, we are led to the system of equations

$$\frac{m_k}{2} = \frac{1}{n} S_k, \qquad k = 0, 1, \cdots, p, \tag{1'}$$

where

$$S_k = \sum_{i=1}^{n} x_i^k, \qquad k = 0, 1, \cdots, p, \cdots.$$

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We observe that (1′) defines the sum of the first p powers of the n numbers $x_1, \cdots, x_n$. Consequently neither $S_{p+1}, \cdots, S_n$ nor the $p+1$st through nth symmetric function of $x_1, \cdots, x_n$ are uniquely determined. Thus if ${}_0T_n$ and ${}_{n-p}T_n$ are nth degree polynomials whose zeros give a set of nodes for $p = n$ and $p < n$ respectively, then

$$_{n-p}T_n = {}_0T_n - \pi_{n-p-1}$$

where π_{n-p-1} is an arbitrary $(n-p-1)$st degree polynomial.** In order to characterize π more exactly, we use a modification of a technique in Hildebrand [4] originally due to Chebyshev. If $|x| > \{1, |x_i|\}$,

$$\int_{-1}^{1} \frac{dt}{x-t} - \frac{2}{n}\frac{_{n-p}T_n'(x)}{_{n-p}T_n(x)} = \sum_{k=p+2}^{\infty} \frac{\mathfrak{a}_{k-1}}{x^k},$$

$$\mathfrak{a}_k = m_k - \frac{2}{n} S_k.$$

After an integration with respect to x and some manipulation we get

$$_{n-p}T_n(x) = cx^n \exp\left\{-n\left[\frac{1}{2\cdot 3x^2} + \frac{1}{4\cdot 5x^4} + \cdots\right] - \left[\frac{b_{p+1}}{x^{p+1}} + \cdots + \frac{b_{n-1}}{x^{n-1}} + \frac{b_n}{x^n}\right]\right\}$$

$$\cdot \exp\left\{-\sum_{k=n+1}^{\infty} \frac{b_k}{x^k}\right\}, \qquad b_k = -\frac{n}{2}\frac{\mathfrak{a}_k}{k}.$$

Because of the left-hand side, series expansion on the right must terminate and thus neither the second exponential nor terms in first past $(1/n(n+1))(1/x^n)$ can contribute to the polynomial part.

Thus

$$_{n-p}T_n(x) = \text{Polynomial Part}\left[cx^n \exp\left\{-n\sum_{j=2}^{n}\frac{c_j}{x^j} - \sum_{j=p+1}^{n}\frac{b_j}{x^j}\right\}\right]$$

$$c_j = \frac{1}{j(j+1)} \qquad j \text{ even},$$

$$= 0 \qquad j \text{ odd}.$$

For $p = n-1$ the contribution of $-b_n/x^n$ appears in the constant term of ${}_1T_n$ as

$$(2) \qquad {}_1T_n = {}_0T_n - b_n = {}_0T_n - S_n/n + m_n/2.$$

For $p = n-2$ the contribution of $-b_{n-1}/x^{n-1} - b_n/x^n$ appears as

$$(3) \qquad {}_1T_n = {}_0T_n - xb_{n-1} - b_n = x\frac{n}{n-1}\left(\frac{m_{n-1}}{2} - \frac{S_{n-1}}{n}\right) - \frac{S_n}{n} + \frac{m_n}{2}.$$

For $n = 8$ or $n = 10$ examination of the curves ${}_0T_n - b_n$ reveals that only for a small range of values of b_n does ${}_0T_n - b_n$ have n real zeros. In that case we have from (2)

$$\sigma_n = (2b_n)^2, \qquad n = 8 \quad \text{or} \quad 10.$$

**$\pi_{-1} \equiv 0$.

For $n = 11$, ${}_0T_{11}$ is odd and has three real zeros. Thus ${}_0T_{11} - b_{11}$ cannot have all real zeros, eliminating the possibility of $p = n - 1$ for this case. If $p = n - 2 = 9$, ${}_0T_{11} - \alpha x - \beta$ will have 11 real zeros for appropriate α and β. From (3)

$$\sigma_{11} = \left[\frac{2}{11}\sum_{j=1}^{11} x_j^{10} - m_{10}\right]^2 + \left[\frac{2}{11}\sum_{j=1}^{11} x_j^{11} - m_{11}\right]^2 \tag{4}$$
$$= \left(\frac{20}{11}\alpha\right)^2 + (2\beta)^2.$$

To preserve symmetry we set $\beta = 0$.

TABLE I

n	b_n	$\sigma_n(b_n)$	$\pm$ Nodes (b_n)	${}_{\min}\sigma_n$
8	-1.01117×10^{-3}	4.08986×10^{-6}	0.	$.79221 \times 10^{-6}$
			.443754	
	$(-.27075 \times 10^{-2} < b_8 < -.101117 \times 10^{-2}$		.572618	
	for real nodes$)$		.899179	
10	-5.955352×10^{-4}	1.41865×10^{-6}	.196220	$.30362 \times 10^{-6}$
			.196124	
	$(-.7655 \times 10^{-3} < b_{10} < -.5955352 \times 10^{-3}$		.571377	
	for real nodes$)$		.920199	
			.645338	
11	α	$\sigma_n(\alpha)$	Nodes (α)	
	-3.14483×10^{-4}	3.26941×10^{-7}	0.0	$.34535 \times 10^{\;6}$*
			$\pm .264246$	
	$(-8.53270 \times 10^{-4} < \alpha < -3.14483 \times 10^{-4}$		$\pm .264492$	
	for real nodes$)$		$\pm .614765$	
			$\pm .674800$	
			$\pm .927502$	

In Table I we list the approximate values of b_8, b_{10} and α that give all real zeros, the minimum σ_n's consistent with these values and the corresponding quadrature nodes. Also we list the minimum σ_n's as computed by Barnhill et al. It should be noted that their minimum σ_{11} is in error and the quadrature formula they have obtained corresponds roughly to selecting an α at the wrong end of the allowable interval since (4) shows $\sigma_{11} \sim \alpha^2$. Also note that for $n = 8$ and $n = 10$ requiring (1) to be exact for $(n - 1)$st degree polynomials only increases σ_n above the computed minimum about a factor of five.

*In error. Nodes corresponding to this value of σ_n are [1]; $\pm.92676$, $\pm.70492$, $\pm.51792$, $\pm.45740$, 0, 0, 0. Compare with nodes corresponding to $\alpha = -8.53270 \times 10^{-4}$ and $\sigma_n(\alpha) = 2.40684 \times 10^{-4}$; $\pm.925039$, $\pm.898403$, $\pm.716634$, $\pm.477831$, $\pm.477155$, 0.

Barnhill observes that the quadrature formula corresponding to minimum σ_n has multiple nodes at the origin. It is no longer clear if this will hold for $n = 11$. By the manner in which they were chosen our quadrature formulas should have multiple nodes at some point in $[-1, 1]$ although for $n = 10$ and 11 we have only computed these nodes so that they agree to three figures.

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