Error Bounds for Polynomial Spline Interpolation*

By Martin H. Schultz

Abstract. New upper and lower bounds for the $L^2$ and $L^m$ norms of derivatives of the error in polynomial spline interpolation are derived. These results improve corresponding results of Ahlberg, Nilson, and Walsh, cf. [1], and Schultz and Varga, cf. [5].

1. Introduction. In this paper, we derive new bounds for the $L^2$ and $L^m$ norms of derivatives of the error in polynomial spline interpolation. These bounds improve and generalize the known error bounds, cf. [1] and [5], in the following important ways: (1) these bounds can be explicitly calculated and are not merely asymptotic error bounds such as those given in [1] and [5]; (2) explicit lower bounds are given for the error for a class of functions; (3) the degree of regularity required of the function, $f$, being interpolated is extended, i.e., in [1] and [5] we demand that the $m$th or $2m$th derivative of $f$ be in $L^2$, if we are interpolating by splines of degree $2m - 1$, while here we demand only that some $p$th derivative of $f$, where $m \leq p \leq 2m$, be in $L^2$; and (4) bounds are given for high-order derivatives of the interpolation errors.

2. Notations. Let $-\infty < a < b < \infty$ and for each positive integer, $m$, let $K^m[a, b]$ denote the collection of all real-valued functions $u(x)$ defined on $[a, b]$ such that $u \in C^m[a, b]$ and such that $D^{m-1}u$ is absolutely continuous, with $D^m u \in L^2[a, b]$, where $Du = du/dx$ denotes the derivative of $u$. For each nonnegative integer, $M$, let $\mathcal{P}(a, b)$ denote the set of all partitions, $\Delta$, of $[a, b]$ of the form

$$\Delta: a = x_0 < x_1 < \cdots < x_M < x_{M+1} = b.$$ 

Moreover, let $\mathcal{P}_M(a, b) = \bigcup_{M=0}^\infty \mathcal{P}_M(a, b)$.

If $\Delta \in \mathcal{P}_m(a, b)$, $m$ is a positive integer and $z$ is an integer such that $m - 1 \leq z \leq 2m - 2$, we define the spline space, $S(2m - 1, \Delta, z)$, to be the set of all real-valued functions $s(x) \in C^m[a, b]$ such that on each subinterval $(x_i, x_{i+1})$, $0 \leq i \leq M$, $s(x)$ is a polynomial of degree $2m - 1$. We remark that our definition is identical with the definition of deficient splines of [1]. For generalizations of this concept of spline subspace, the reader is referred to [5]. In particular, it is easy to verify that all the results of this paper remain essentially unchanged if one allows the number $z$ to depend on the partition points, $x_i, 1 \leq i \leq M$, in such a way that $m - 1 \leq z(x_i) \leq 2m - 2$ for all $1 \leq i \leq M$. The details are left to the reader.

Following [1] we define the interpolation mapping $s_m: C^{m-1}[a, b] \to S(2m - 1, \Delta, z)$ by $s_m(f) = s$, where

$$D^k s(x_i) \equiv D^k f(x_i), \quad 0 \leq k \leq 2m - 2 - z, \quad 1 \leq i \leq M,$$

$$0 \leq k \leq m - 1, \quad i = 0 \text{ and } M + 1.$$ 

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507
We remark that the preceding interpolation mapping corresponds to the Type I interpolation of [1]. It is easy to modify the results of this paper for the cases in which the interpolation mapping corresponds to Types II, III, and IV interpolation of [1]. The details are left to the reader.

3. Basic $L^2$-Error Bounds. In this section, we obtain explicit upper and lower bounds for the quantities $\Lambda(m, p, z, j)$, $1 \leq m, m \leq p \leq 2m, m - 1 \leq z \leq 2m - 2$, and $0 \leq j \leq m$, defined by

$$\Lambda(m, p, z, j) = \sup \{ \| D^j(f - \delta_m f) \|_{L^2[a, b]} / \| D^p f \|_{L^2[a, b]} \mid f \in K^p[a, b], \| D^p f \|_{L^2[a, b]} \neq 0 \}.$$  

First, we recall some basic results from [1] and [5] and introduce some additional notation.

**Theorem 3.1.** The interpolation mapping given by (2.2) is well defined for all $\Delta \in \Theta(a, b)$, $1 \leq m$, and $m - 1 \leq z \leq 2m - 2$.

**Theorem 3.2 (First Integral Relation).** If $f \in K^m[a, b]$, $1 \leq m$, $\Delta \in \Theta(a, b)$, and $m - 1 \leq z \leq 2m - 2$,

$$\| D^m f \|_{L^2[a, b]} = \| D^m(f - \delta_m f) \|_{L^2[a, b]} + \| D^m \delta_m f \|_{L^2[a, b]}.$$  

**Theorem 3.3 (Second Integral Relation).** If $f \in K^{2m}[a, b]$, $1 \leq m$, $\Delta \in \Theta(a, b)$, and $m - 1 \leq z \leq 2m - 2$,

$$\| D^m(f - \delta_m f) \|_{L^2[a, b]} = \int_a^b (f - \delta_m f) D^{2m} f \, dx.$$  

Finally, following Kolmogorov, cf. [4, p. 146], if $t$ and $d$ are positive integers, let $\lambda_d(t)$ denote the $d$th eigenvalue of the boundary value problem,

$$(-1)^d D^{2t} y(x) = \lambda_d x, \quad 0 < x < b,$$

$$D^t y(a) = D^t y(b) = 0, \quad 0 \leq k \leq 2t - 1,$$

where the $\lambda_d$ are arranged in order of increasing magnitude and repeated according to their multiplicity. We remark that the problem (3.4)–(3.5) has a countably infinite number of eigenvalues, all of which are nonnegative and it may be shown that

$$\lambda_d = (\pi/(b - a))^{2t} d^{2t}[1 + O(d^{-1})], \quad \text{as } t < d \to \infty.$$  

Using the bootstrapping technique of [1, p. 92], and letting

$$\Delta \equiv \max \{(x_{i+1} - x_i) \mid 0 \leq i \leq M\} \quad \text{and} \quad \Delta \equiv \min \{(x_{i+1} - x_i) \mid 0 \leq i \leq M\},$$

for all $\Delta \in \Theta_m(a, b)$, we have the following generalization of Theorem 7 of [5].

**Theorem 3.4.**

$$\lambda_d^{-1/2}(m - j) \leq \Lambda(m, m, z, j) \leq K_{m, z, j}(\Delta)^{m-j},$$

where

$$d \equiv (M + 1)(2m - z + 1) + z - j + 2.$$
and
\[ K_{m,m,z,i} = 1, \quad \text{if } m - 1 \leq z \leq 2m - 2, j = m, \]
\[ = \left(\frac{1}{\pi}\right)^{m-j}, \quad \text{if } m - 1 = z, 0 \leq j \leq m - 1, \]
\[ = \frac{(z + 2 - m)!}{m! \pi^{m-j}}, \quad \text{if } m - 1 \leq z \leq 2m - 2, 0 \leq j \leq 2m - 2 - z, \]
\[ = \frac{(z + 2 - m)!}{j! \pi^{m-j}}, \quad \text{if } m - 1 \leq z \leq 2m - 2, 2m - 2 - z \leq j \leq m - 1, \]

for all \( 1 \leq m, 0 \leq M, \Delta \subseteq \mathcal{P}_d(a, b), m - 1 \leq z \leq 2m - 2, \text{ and } 0 \leq j \leq m. \)

Proof. First, we prove the right-hand inequality of (3.6). If \( m - 1 \leq z \leq 2m - 2 \) and \( j = m \), the result follows directly from Theorem 3.2.

Otherwise, \( D^t(f - s_m f)(x_i) = 0, 1 \leq i \leq M, 0 \leq j \leq 2m - 2 - z, \) and by the Rayleigh-Ritz inequality, cf. [3, p. 184],
\[ \int_{x_i}^{x_{i+1}} (D^t(f - s_m f)(x))^2 \, dx \leq \left(\frac{\Delta}{\pi}\right)^2 \int_{x_i}^{x_{i+1}} (D^{t+1}(f - s_m f)(x))^2 \, dx, \]
\[ 0 \leq j \leq 2m - 2 - z. \] Summing both sides of (3.9) with respect to \( i \) from 0 to \( M \), we obtain
\[ \|D^t(f - s_m f)\|_{L^2[a,b]} \leq \left(\frac{\Delta}{\pi}\right)^2 \|D^{t+1}(f - s_m f)\|_{L^2[a,b]}, \]
\[ 0 \leq j \leq 2m - 2 - z. \] Using (3.10) repeatedly we obtain
\[ \|D^t(f - s_m f)\|_{L^2[a,b]} \leq \left(\frac{\Delta}{\pi}\right)^{m-1-z-i} \|D^{m}(f - s_m f)\|_{L^2[a,b]}, \]
Hence, if \( 2m - 1 - z = m \), i.e., \( z = m - 1 \), then
\[ \|D^t(f - s_m f)\|_{L^2[a,b]} \leq \left(\frac{1}{\pi}\right)^{m-i} (\Delta)^{-z} \|D^{m}(f - s_m f)\|_{L^2[a,b]}, \]
which is the required result for this special case.

Otherwise, since \( m \leq z \), applying Rolle's Theorem to \( D^{m-2-i}(f - s_m f) \in C^{r-m+i}[a, b] \), which vanishes at every mesh point, we have that for each \( 0 \leq j \leq z - m + 1, \) there exist points \( \{\xi_l^{(i)}\}_{l=0}^{m-i} \) in \( [a, b] \) such that
\[ D^{m-2-i}(f - s_m f)(\xi_l^{(i)}) = 0, \quad 0 \leq j \leq m - 1 - (2m - 2 - z), \]
\[ = z - m + 1, \quad 0 \leq l \leq M + 1 - j, \]
\[ a = \xi_0^{(i)} < \xi_1^{(i)} \cdots < \xi_{s+1-i}^{(i)} = b, \quad 0 \leq j \leq z - m + 1, \]
\[ \xi_l^{(i)} \leq \xi_l^{(i+1)} < \xi_{l+1}^{(i)}, \quad \text{for all } 0 \leq l \leq M + 1 - j, 0 \leq j \leq z - m + 1 \]
and
\[ |\xi_{l+1}^{(i)} - \xi_l^{(i)}| \leq (j + 1)\Delta, \quad 0 \leq l \leq M - j, 0 \leq j \leq z - m + 1, \]
i.e., choose \( \xi_l^{(0)} = x_l, 0 \leq l \leq M + 1. \)
Thus, applying the Rayleigh-Ritz inequality, we have
\[
\int_{\xi_{l+1}(t)}^{\xi_{l+1}(t)} (D^{m-2-\varepsilon+(\varepsilon+1)}(f - \sigma_m))(x)^2 \, dx
\]
(3.17)
\[
\leq \left( \frac{(j+1)\Delta}{\pi} \right)^2 \int_{\xi_{l+1}(t)}^{\xi_{l+1}(t)} (D^{m-2-\varepsilon+(\varepsilon+1)}(f - \sigma_m))^2 \, dx
\]
for all \(0 \leq l \leq M - j, 0 \leq j \leq z - m + 1\). Summing (3.17) with respect to \(l\) from 0 to \(M - j\), we have
\[
||D^{m-2-\varepsilon+(\varepsilon+1)}(f - \sigma_m)||_{L^2_{[a,b]}} \leq \frac{(j+1)\Delta}{\pi} ||D^{m-2-\varepsilon+(\varepsilon+1)}(f - \sigma_m)||_{L^2_{[a,b]}},
\]
(3.18)
\[0 \leq j \leq z - m + 1\). Using (3.18) repeatedly along with (3.2) we have
\[
||D^{m-1-\varepsilon}(f - \sigma_m)||_{L^2_{[a,b]}} \leq \frac{(z+2-m)!}{\pi^{m-1}} (\Delta)^{m-1} ||D^{m-1}(f - \sigma_m)||_{L^2_{[a,b]}},
\]
(3.19)
Combining (3.11) with (3.19), we have that
\[
||D^{t'}(f - \sigma_m)||_{L^2_{[a,b]}} \leq \frac{(z+2-m)!}{\pi^{m-1}} (\Delta)^{m-1} ||D^{m-1}(f - \sigma_m)||_{L^2_{[a,b]}},
\]
(3.20)
if \(0 \leq j \leq 2m - 2 - z\). Otherwise, it follows from (3.18) that
\[
||D^{t'}(f - \sigma_m)||_{L^2_{[a,b]}} \leq \frac{(z+2-m)!}{j! \pi^{m-1}} ||D^{m-1}(f - \sigma_m)||_{L^2_{[a,b]}},
\]
(3.21)
Finally, we prove the left-hand inequality of (3.6). This inequality follows directly from a fundamental result of Kolmogorov, cf. [4, p. 146], which states that
\[
\lambda_{t+1}^{-1/2}(m - j) \leq \Delta(m, m, z, j),
\]
(3.22)
where \(t \equiv \text{dimension } D'(S(2m - 1, \Delta, z))\), for all \(1 \leq m, 0 \leq M, \Delta \in \partial_{s}(a, b), m - 1 \leq z \leq 2m - 2\), and \(0 \leq j \leq m\). But the space \(D'(S(2m - 1, \Delta, z))\) has dimension \(t \equiv (2m-j)(M+1) - (z+j-1)M = (M+1)(2m-z+1) + z-j+1\).
Q.E.D.

We remark that in this case it is easy to verify that there exists a positive constant, \(K\), such that
\[
\lambda_{t+1/2}^{-1/2}(2m - j) \leq \Delta(m, 2m, z, j) \leq K_{m, z} \Delta^{m-1}/(M + 1)
\]
(3.23)
where \(s = (2m - z + 1 + (z - j + 2)/(M + 1))\), and thus that splines are "quasi-optimal".

The next result generalizes Theorem 9 of [5].

THEOREM 3.5.
where

\[ d = (M + 1)(2m - z + 1) + z - j + 2 \]

and

\[ K_{m,m,z,j} = (K_{m,m,z,j})(K_{m,m,z,0}), \quad \text{for all } 1 \leq m, 0 \leq M, \Delta \in \mathcal{O}_M(a, b), \]

\[ m - 1 \leq z \leq 2m - 2, \quad \text{and } 0 \leq j \leq m. \]

**Proof.** Applying the Cauchy-Schwarz inequality to the Second Integral Relation yields the inequality

\[ ||D^m(f - g_m)||_{L^1[a,b]} \leq ||D^{2m}||_{L^1[a,b]}||f - g_m||_{L^2[a,b]}. \]

Applying the proof of Theorem 3.4, we have

\[ ||D^j(f - g_m)||_{L^1[a,b]} \leq K_{m,m,z,j}||D^m(f - g_m)||_{L^1[a,b]}(\Delta)^{-j}. \]

Using (3.27) for the special case of \( j = 0 \) in (3.26) yields

\[ ||D^m(f - g_m)||_{L^1[a,b]} \leq ||D^{2m}||_{L^1[a,b]} K_{m,m,z,0}(\Delta)^{m}. \]

Using (3.28) to bound the right-hand side of (3.27) gives us the right-hand inequality of (3.23). The left-hand inequality of (3.23) follows as in Theorem 3.4. Q.E.D.

We now recall a fundamental inequality of E. Schmidt which will be used several times in the remainder of this paper.

**Lemma 3.1.** If \( p_N(x) \) is a polynomial of degree \( N \),

\[ ||Dp_N||_{L^1[a,b]} \leq \frac{E_N}{b - a} ||p_N||_{L^1[a,b]}, \]

where \( E_N \equiv (N + 1)^2 \sqrt{2}. \)

**Proof.** Cf. [2]. Q.E.D.

**Theorem 3.6.**

\[ \lambda_d^{-1/2}(p - j) \leq \Delta(m, p, z, j) \leq K_{m,p,z,j}(\Delta)^{p-j}, \]

where

\[ d = (M + 1)(2m - z + 1) + z - j + 2 \]

and

\[ K_{m,p,z,j} = \left\{ K_{p,p,2m-1,j} + K_{m,m,2m,z,j}, 2^{(1/2)(2m-p)} \left[ \frac{p!}{(2p - 2m)!} \right]^2 (\Delta/\Delta)^{2m-p} \right\} \]

for all \( 1 \leq m, 0 \leq M, \Delta \in \mathcal{O}_M(a, b), m < p < 2m, 4m - 2p - 1 \leq z \leq 2m - 2, \) and \( 0 \leq j \leq m. \)

**Proof.** Consider \( S(2p - 1, \Delta, 2m - 1) \subset K^{2m}[a,b]. \) This space is well defined since \( 2p - 2 \geq 2(m + 1) - 2 = 2m. \) Moreover, if \( s_m \) denotes the interpolation mapping of \( C^{m-1}[a, b] \) into \( S(2m - 1, \Delta, z) \) and \( s_p \) denotes the interpolation mapping of \( C^{m-1}[a, b] \) into \( S(2p - 1, \Delta, 2m - 1), \) then \( s_m(s_p f) = s_{mp} f \) for all \( f \in C^{m-1}[a, b]. \)

In fact, \( D^i s_f \) interpolates \( D^i f \) at \( \xi_i, 1 \leq i \leq M, \) for all \( 0 \leq k \leq 2p - (2m - 1) - 2 = 2p - 2m - 1, \) while \( D^i s_{mp} f \) interpolates \( D^i f \) at \( \xi_i, 1 \leq i \leq M, \) for all \( 0 \leq k \leq 2m - z - 2 = 2m - 2 - (4m - 2p - 1) - 2 = 2p - 2m - 1. \)
Thus,
\begin{equation}
||D^j(f - \sigma_m f)||_{L^1(a, b)} \leq ||D^j(f - \sigma_p f)||_{L^1(a, b)} + ||D^j(\sigma_p f - \sigma_m(\sigma_p f))||_{L^1(a, b)}, \quad 0 \leq j \leq m.
\end{equation}

By Theorem 3.4,
\begin{equation}
||D^j(f - \sigma_p f)||_{L^1(a, b)} \leq K_{p, p, 2m-1, i}(\Delta)^{p-j}||D^p f||_{L^1(a, b)},
\end{equation}
and by Theorem 3.5
\begin{equation}
||D^j(\sigma_p f - \sigma_m(\sigma_p f))||_{L^1(a, b)} \leq K_m(2m, z, i)(\Delta)^{2m-j}||D^{2m} \sigma_p f||_{L^1(a, b)}.
\end{equation}

But by Schmidt's inequality and the First Integral Relation, since \( \sigma_p f \) is a piecewise polynomial of degree \(2p-1\) with \(p > m\), we have
\begin{equation}
||D^m \sigma_p f||_{L^1(a, b)} \leq \left( \sum_{i=1}^{2m-p} \frac{E_{2p-2m-1+i}}{(2p-2m-1+i)!} \right) \frac{||D^p f||_{L^1(a, b)}}{(\Delta)^{2m-p}}.
\end{equation}

The required result now follows from (3.33), (3.34), (3.35), and (3.36). Q.E.D.

4. \( L^2 \)-Error Bounds for Higher Order Derivatives. In this section we give explicit upper bounds for the quantities \( \Lambda(m, p, z, j) \) in the special cases of \( m < p - 2m \) and \( m < j \leq p \). Since \( \sigma_m f \) is not necessarily in \( K'[a, b] \) if \( z + 1 < j \leq p \), it is necessary to modify the definition of \( \Lambda(m, p, z, j) \) given in (3.1). The new definition is given by
\begin{equation}
\Lambda(m, p, z, j) \equiv \sup \left\{ \left( \sum_{i=0}^{M} \frac{||D^i(f - \sigma_m f)||_{L^1(a, b)}^2}{||D^p f||_{L^1(a, b)}} \right)^{1/2} \right\}.
\end{equation}

The main result of this section is
THEOREM 4.1.
\begin{equation}
\Lambda(m, p, z, j) \leq K_{m, p, z, i}(\Delta)^{p-j},
\end{equation}
where
\begin{equation}
K_{m, p, z, i} \equiv \left[ K_{p, p, z, i} + (K_{m, p, z, m} + K_{p, p, p, m})i^{-j} \right] \frac{(2p + m)!}{(2p - j)!} \left( \frac{\Delta}{\Delta} \right)^{i-m},
\end{equation}
for all \( 1 \leq m, 0 \leq M, \Delta \in \Phi_M(a, b), m < p \leq 2m, 4m - 2p - 1 \leq z \leq 2m - 2 \), and \( m < j \leq p \).

Proof. By Theorem 3.6,
\begin{equation}
||D^m(f - \sigma_m f)||_{L^1(a, b)} \leq K_{m, p, z, m}(\Delta)^{p-m},
\end{equation}
and by Theorem 3.4,
\begin{equation}
||D^k(f - \sigma_p f)||_{L^1(a, b)} \leq K_{p, p, p, k}(\Delta)^{p-k}, \quad 0 \leq k \leq p.
\end{equation}
Combining (4.4) and (4.5), we obtain

\[ ||D^m(s_mf - s_pf)||_{L^1[a, b]} \leq (K_{m, p, z, m} + K_{p, p, p, m})(\Delta)^{m-1}. \]

Using the Schmidt inequality in (4.6), we obtain

\[ ||D^i(s_mf - s_pf)||_{L^1[a, b]} \leq \frac{\left( \prod_{i=1}^{m-1} E_{(2p-1)-i+1} \right)}{(\Delta)^{i-1-m}} ||D^m(s_mf - s_pf)||_{L^1[a, b]} \]

\[ \leq (K_{m, p, z, m} + K_{p, p, p, m})(\prod_{i=1}^{m-1} E_{(2p-1)-i+1})(\Delta)^{-i}(\Delta/\Delta)^{-i}. \]

The required result follows from (4.5), (4.7), and

\[ ||D^i(f - s_mf)||_{L^1[a, b]} \leq ||D^i(f - s_pf)||_{L^1[a, b]} + ||D^i(s_pf - s_mf)||_{L^1[a, b]}. \]

Q.E.D.

We remark that in those cases in which \( s_mf \in K'[a, b] \), lower bounds of the form introduced in Section 3 can be given for \( \Lambda(m, p, z, j) \).

5. \( L^m \)-Error Bounds. In this section, we give explicit upper bounds for the quantities \( \Lambda^m(m, p, z, j) \), \( 1 \leq m, m \leq p \leq 2m, m - 1 \leq z \leq 2m - 2 \), and \( 0 \leq j \leq p \), defined by

\[ \Lambda^m(m, p, z, j) = \text{Sup} \left\{ \max_{0 \leq i \leq M} \left( ||D^i(f - s_mf)||_{L^m([z, z+1])}/||D^j||_{L^1[a, b]} \right) \middle| f \in K^m[a, b], ||D^j||_{L^1[a, b]} \neq 0 \right\}. \]

We obtain the following results as corollaries of the results of Section 3 and Section 4. As an improvement of Theorem 6 of [5], we have

**Theorem 5.1.**

\[ \Lambda^m(m, m, z, j) \leq K^m_{m, m, z, i}(\Delta)^{m-j-1/2}, \]

where

\[ K^m_{m, m, z, i} = K_{m, m, z, i+1}, \quad \text{if} \quad m - 1 = z, 0 \leq j \leq m - 1, \]

\[ K^m_{m, m, z, i+1}, \quad \text{if} \quad m - 1 < z \leq 2m - 2, 0 \leq j \leq 2m - 2 - z, \]

\[ (j - 2m + 3 + z)^{1/2} K^m_{m, m, z, j+1}, \quad \text{if} \quad m - 1 < z \leq 2m - 2, \]

\[ 2m - 2 - z < j \leq m - 1, \]

for all \( 1 \leq m, 0 \leq M, \Delta \in \mathcal{D}_{L}(a, b) \), \( m - 1 \leq z \leq 2m - 2 \), and \( 0 \leq j \leq m - 1 \).

**Proof.** We give the proof in the special case of \( m - 1 = z \), \( 0 \leq j \leq m - 1 \), as the proof in the other cases is analogous. Given any \( x \in [a, b] \), there exists a point \( y \in [a, b] \) such that \( D^i(f - s_mf)(y) = 0 \) and \( |x - y| \leq \Delta \). Hence, \( D^i(f - s_mf)(x) = \int_x^y D^{i+1}(f - s_mf)(t) \, dt \) and

\[ ||D^i(f - s_mf)||_{L^m([a, b])} \leq (\Delta)^{i/2} ||D^{i+1}(f - s_mf)||_{L^1[a, b]}. \]

The result now follows from applying Theorem 3.4 to the right-hand side of the preceding inequality. Q.E.D.
As in Theorem 5.1, we have as an improvement of Theorem 8 of [5].

**Theorem 5.2.**

\[ \Lambda^m(m, 2m, z, j) \leq K^m_{2m, z, i}(\Delta)^{2m-i-1/2}, \]

where

\[ K^m_{2m, z, i+1} = K^m_{2m, z, i+1}, \text{ if } m-1 = z, 0 < j \leq m-1, \]
\[ = K^m_{2m, z, i+1}, \text{ if } m-1 < z \leq 2m-2, 0 \leq j \leq 2m-2-z, \]
\[ = (j - 2m + 3 + z)^{1/2} K^m_{2m, z, i+1}, \text{ if } m-1 < z \leq 2m-2, \]
\[ 2m-2-z < j \leq m-1, \]

for all \(1 < m, 0 \leq M, \Delta \in \mathcal{O}(a, b), m-1 \leq z \leq 2m-2, \) and \(0 \leq j \leq m-1.\)

As in Theorem 3.6, we have

**Theorem 5.3.**

\[ \Lambda^m(m, p, z, j) \leq K^m_{p, z, i}(\Delta)^{p-i-1/2}, \]

where

\[ K^m_{p, z, i} = \left\{ K^m_{p, 2m-1, i} + K^m_{p, 2m, i} \cdot 2^{(2m-p)/2} \left( \frac{p!}{(2p-2m)!} \right) \left( \frac{\Delta}{\Delta} \right)^{2m-p} \right\}, \]

for all \(1 \leq m, 0 \leq M, \Delta \in \mathcal{O}(a, b), m < p < 2m, 4m - 2p - 1 \leq z \leq 2m - 2,\)

and \(0 \leq j \leq m-1.\)

Finally, to give a result analogous to Theorem 4.1, we need an inequality due to A. A. Markov.

**Lemma 5.1.** If \(p_N(x)\) is a polynomial of degree \(N,\) then

\[ ||D^m p_N||_{L^m[a,b]} \leq \frac{M_N}{b-a} ||p_N||_{L^m[a,b]}, \]

where \(M_N = 2N^2.\)

**Proof.** Cf. [6]. Q.E.D.

As an extension of Theorem 10 of [5], we prove

**Theorem 5.4.**

\[ \Lambda^m(m, p, z, j) \leq K^m_{p, z, i}(\Delta)^{p-i-1/2}, \]

where

\[ K^m_{p, z, i} = \left\{ K^m_{p, p, i} + (K^m_{p, p, i} + K^m_{p, p, i}) \cdot 2^{i-m+1} \left( \frac{(2p-m)!}{(2p-j-1)!} \right) \left( \frac{\Delta}{\Delta} \right)^{i-m+1} \right\}, \]

for all \(1 \leq m, 0 \leq M, \Delta \in \mathcal{O}(a, b), m < p \leq 2m, 4m - 2p - 1 \leq z \leq 2m - 2,\)

and \(m \leq j \leq p-1.\)

**Proof.** From Theorem 5.1, we have that

\[ ||D^k(f - g)\|_{L^m[a,b]} \leq K^m_{p, p, a}(\Delta)^{p-k-1/2} ||D^k f\|_{L^m[a,b]}, 0 \leq k \leq p-1, \]

and from Theorem 5.3

\[ ||D^{m-1}(f - g)\|_{L^m[a,b]} \leq K^m_{p, p, m-1}(\Delta)^{p-m+1/2} ||D^m f\|_{L^m[a,b]}. \]
Combining (5.11) and (5.12), we have

\[(5.13) \quad \| D^{m-1}(\xi_f - \xi_p) \|_{L^\infty[a,b]} \leq (K_{m,p,p+1}^m + K_{p,p+1}^m)(\Delta)^{p-m+1/2}\| D^p f \|_{L^1[a,b]}. \]

But,

\[(5.14) \quad \| D^i(\xi_f - \xi_p) \|_{L^\infty[a,b]} \leq \frac{\left( \prod_{i=1}^{j-1} M_{2p-1-i} \right)}{(\Delta)^{j-1}} \| D^{m-1}(\xi_f - \xi_p) \|_{L^\infty[a,b]} \leq 2^{j-1}(\Delta)^{j-m+1} \cdot \| D^{m-1}(\xi_f - \xi_p) \|_{L^\infty[a,b]}, \]

where

\[ \| \cdot \|_{L^\infty[a,b]} = \max_{0 \leq t \leq 1} \| \cdot \|_{L^\infty[\alpha, \beta[}. \]

The required result follows directly from (5.11), (5.13), (5.14), and the observation that

\[ \| D^i(f - \xi_f) \|_{L^\infty[a,b]} \leq \| D^i(f - \xi_p) \|_{L^\infty[a,b]} + \| D^i(\xi_p - \xi_f) \|_{L^\infty[a,b]}. \]

Q.E.D.

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