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Some Observations on Interpolation in Higher Dimensions

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Abstract. This paper presents a method, based on successive displacement of the coordinates, both for finding suitable interpolation points and for constructing the interpolating polynomial for functions of more than one independent variable.

1. Introduction. The problem of interpolation, in particular polynomial interpolation, in higher dimensions has received some attention in recent years, [1]–[9], but except in certain special cases, [4], [5], [7], the algorithms for computation are complicated and difficult to use. We shall restate the interpolation problem and give a method for solving it in the case when the interpolating function is a polynomial of specified total degree.

2. The Problem. Let R_n denote n-dimensional Euclidean space and let $x = (x_1, \cdots, x_n)$ denote a generic point in R_n. Let $S \subset R_n$ be a set with a nonempty interior. Let $x^{(1)}, \cdots, x^{(m)}$ be m points in S. The simplest interpolation problem consists in finding a set of linearly independent functions, $f_1, \cdots, f_m$, defined on S and constants $a_1, \cdots, a_m$ so that the function

$$\sum_{i=1}^{m} a_i f_i(x) \tag{2.1}$$

coincides with the values of a function f at the points $x^{(i)}$. The functions $f_1, \cdots, f_m$ should satisfy certain computational requirements and, ideally, resemble the function being interpolated in global behavior.

On the other hand, precisely because of computational and other considerations, it is often desirable to specify first the m linearly independent functions $f_1, \cdots, f_m$. One then forms the expression (2.1). To be able to determine the constants a_i uniquely from the requirement that the function (2.1) takes on prescribed values at the points $x^{(i)}$, it is necessary and sufficient that the determinant of the matrix $(f_i(x^{(i)}))$ be different from zero. We shall refer to m points in S such that the determinant of $(f_i(x^{(i)}))$ is different from zero as interpolation points for the functions $f_1, \cdots, f_m$, or simply as interpolation points when it is clear to what set of functions we are referring. The general question of the existence of interpolation points for a given set of linearly independent functions may be answered in the affirmative using an argument of Davis [3].

We shall treat the special case of finding interpolation points so that the polynomial of degree l,

$$p_l(x) = \sum a_{\alpha_1 \cdots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

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the summation taken over all the α_i, $\alpha_i = 0, 1, \cdots, l$, $i = 1, \cdots, n$, with $\alpha_1 + \cdots + \alpha_n \leqq l$, is uniquely determined by the requirement that it takes on specified values at the interpolation points and we shall give a practical method for determining the $a_{\alpha_1 \cdots \alpha_n}$. For error estimates and methods for determining error estimates, we refer to [7].

In the sequel we shall assume for convenience that S is a sufficiently large open set in R_n which contains the origin.

3. Interpolation Points for Polynomials. We begin with the case $n = 2$, and, as is customary, we set $x_1 = x$, $x_2 = y$. A general lth order polynomial is of the form

$$p_l(x, y) = \sum a_{\mu\nu} x^\mu y^\nu, \tag{3.1}$$

the summation being taken over $\mu, \nu = 0, 1, \cdots, l$ with $\mu + \nu \leqq l$. We give a recursive method for generating the points. For $l = 0$, we may choose any point. For $l = 1$, any three points in general position will serve as interpolation points. In general suppose that in any subdomain of S, we can find interpolation points for polynomials of degree $\leqq l - 1$, $l \geqq 2$. For $p_l(x, y)$ choose $(0, 0)$ and any other l distinct points on the x axis and any other l distinct points on the y axis. Prescribing the values of $p_l(x, y)$ at these points determines the coefficients $a_{00}, a_{10}, \cdots, a_{l0}, a_{01}, \cdots, a_{0l}$. We now write $p_l(x, y)$ in the form

$$p_l(x, y) \equiv a_{00} + \sum_{\mu=1}^{l} a_{\mu 0} x^\mu + \sum_{\nu=1}^{l} a_{0\nu} y^\nu + xy q_{l-2}(x, y),$$

where $q_{l-2}(x, y)$ is a polynomial of degree $l - 2$. By our induction assumption we may find interpolation points for $q_{l-2}(x, y)$ subject to the condition that these points do not lie on the x or y axes. Thus, we may find interpolation points for $p_l(x, y)$.

In the next section we shall show how this recursive procedure can be made into a viable computational method for determining interpolation points and constructing the corresponding interpolating polynomial. It should also be observed that this discussion carries over to the case of polynomials of degree k in x and l in y.

Similar ideas carry over to the n-dimensional case except that the points are chosen recursively with respect to the dimension as well as with respect to the degree. In fact, let us again denote the polynomial of degree l by

$$p_l(x) = \sum a_{\alpha_1 \cdots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{3.2}$$

the sum taken over all the α_i, $i = 1, \cdots, n$, running from 0 to l with $\alpha_1 + \cdots + \alpha_n \leqq l$.

In the case $l = 0$, any point is an interpolation point and in the case $l = 1$, any set of $n + 1$ points in general position is a set of interpolation points. The case $l = 2$ already illustrates the general technique. Suppose we can find interpolation points for polynomials of degree 2 in $n - 1$ dimensions. Let $x' = (x_1, \cdots, x_{n-1}, 0)$. Then $p_2(x')$ is a polynomial of degree 2 in $n - 1$ dimensions so we can find interpolation points for it on the hyperplane $x_n = 0$. Next, observe that $p_2(x)$ can be written in the form

$$p_2(x) \equiv p_2(x') + x_n q_1(x)$$

where $q_1(x)$ is a linear polynomial in n dimensions. To determine the $q_1(x)$, we choose $n + 1$ points in general position off the hyperplane $x_n = 0$. Thus, we arrive at a set of interpolation points for $p_2(x)$.

Generally, suppose we can find interpolation points for polynomials of degree l in $n - 1$ dimensions and degree $l - 1$ in n dimensions in any subdomain of S. Again let $x' = (x_1, \cdots, x_{n-1}, 0)$. Then $p_l(x')$ is a polynomial of degree l in $n - 1$ dimensions and by assumption we can find interpolation points for it. Write $p_l(x)$ in the form

$$p_l(x) = p_l(x') + x_n q_{l-1}(x),$$

where $q_{l-1}(x)$ is a polynomial of degree $l - 1$ in n dimensions. By assumption we may choose interpolation points for $q_{l-1}(x)$ off the hyperplane $x_n = 0$ and thus arrive at a set of interpolation points for $p_l(x)$.

It should be pointed out that the problem of determining interpolation points for polynomials has also been considered to some extent by Thacher and Milne. See [6], [8] and [9]. Their result is that a set of points lying on a simplicial grid, or any nonsingular linear transformation of these points is a suitable set of interpolation points. The recursive procedure discussed above for determining interpolation points for the $p_l(x)$ relies on the fact that such points can be determined for polynomials of degree $\leqq l - 1$ and for polynomials of degree l in $n - 1$ dimensions. Consequently, if one can determine such interpolation points, either recursively as we have suggested, or by the method suggested by Thacher and Milne, or by any other method, the technique described above gives a method for determining interpolation points for $p_l(x)$.

4. Practical Determination of the Interpolating Polynomials. In this section we shall treat the two-dimensional case in detail and simply comment on how to proceed in the n-dimensional case.

Consider again the polynomial (3.1). Suppose that the coefficients $a_{\mu\nu}$ are to be chosen so that $p_l(x, y)$ takes on prescribed values at the interpolation points, the location of which will be specified below. Set first $y = 0$ and choose the origin and l additional distinct points along the x axis. Use the Newton or Lagrange or any other one-dimensional interpolation polynomial to interpolate these points. This determines the coefficients $a_{00}, a_{10}, \cdots, a_{l0}$. Next set $x = 0$ and choose l distinct points along the y axis excluding the origin. Then

$$[p_l(0, y) - a_{00}]/y = \sum_{\nu=1}^{l} a_{0\nu} y^{\nu-1}$$

and the $a_{01}, \cdots, a_{0l}$ can be determined by using a one-dimensional interpolating polynomial.

Having determined $a_{00}, a_{10}, \cdots, a_{l0}, a_{01}, \cdots, a_{0l}$, write (3.1) in the form

$$p_l(x, y) - p_l(x, 0) - p_l(0, y) + a_{00} = xy q_{l-2}(x, y)$$

where

$$q_{l-2}(x, y) = \sum a_{\mu\nu} x^{\mu-1} y^{\nu-1}, \tag{4.1}$$

the summation taken over $\mu, \nu = 1, 2, \cdots, l$ with $2 \leqq \mu + \nu \leqq l$. Next translate the x and y axes and then, possibly, rotate the axes. Thus we must find interpolation points for the polynomial

$$\tilde{q}_{l-2}(\tilde{x}, \tilde{y}) = \sum \tilde{a}_{\mu\nu}\tilde{x}^{\mu-1}\tilde{y}^{\nu-1},$$

the summation taken over $\mu, \nu = 1, 2, \cdots, l$ with $2 \leqq \mu + \nu \leqq l$. Here $\tilde{q}_{l-2}$ is a polynomial of degree $l - 2$ in the $\tilde{x} - \tilde{y}$ coordinate system which is obtained from q_{l-2} after the translation and possible rotation of the coordinates.

We now proceed as before. We set first $\tilde{y} = 0$ and choose the origin in the $\tilde{x} - \tilde{y}$ coordinate system and $l - 2$ additional points on the $\tilde{x}$ axis, none of which lie on the old x or y axes. This allows us to determine the coefficients $\tilde{a}_{11}, \tilde{a}_{21}, \cdots, \tilde{a}_{l-1,1}$. Next we set $\tilde{x} = 0$ and choose $l - 2$ points on the $\tilde{y}$ axis, none of which lie on the old x or y axes and use these points to determine the coefficients $\tilde{a}_{12}, \cdots, \tilde{a}_{1,l-1}$.

Continuing in this way, we are able to determine the polynomial $p_l(x, y)$. The real effectiveness of the algorithm just described may be appreciated by realizing that we simply displace the origin successively, choose appropriate interpolation points along straight lines, and then use the one-dimensional interpolation theory.

A special case which is important for applications is the restriction to a rectangular lattice with h and k the x and y periods, respectively, [2]. From the points $(0, 0)$, $(h, 0)$, $(-h, 0)$, $\cdots$, $([l/2]h, 0)$, $(-[l/2]h, 0)$ if l is even, and if l is odd, the additional point $(([l/2] + 1)h, 0)$, we can determine the coefficients $a_{00}, a_{10}, \cdots, a_{l0}$, and by means of the points $(0, k)$, $(0, -k)$, $\cdots$, $(0, [l/2]k)$, $(0, -[l/2]k)$ if l is even, and if l is odd, the additional point $(0, ([l/2] + 1)k)$, we can determine the coefficients $a_{01}, \cdots, a_{0l}$. Now translate the origin to (h, k) and repeat the process using the points (h, k), $(2h, k)$, $(-h, k)$, $\cdots$, $(-[(l - 2)/2]h, k)$, $(2 + [(l - 2)/2]h, k)$ if l is even, and if l is odd, the additional point $((3 + [(l - 2)/2])h, k)$ and also the points $(h, 2k)$, $(h, -k)$, $\cdots$, $(h, 2 + [(l - 2)/2]k)$, $(h, [(l - 2)/2]k)$ if l is even, and if l is odd, the additional point $(h, (3 + [(l - 2)/2])k)$. In this way the coefficients $a_{11}, a_{21}, \cdots, a_{l-1,1}, a_{12}, \cdots, a_{1,l-1}$ of the polynomial $q_{l-2}(x, y)$ may be determined. Clearly, we may proceed in this way to determine completely the interpolating polynomial $p_l(x, y)$ using only points from the lattice $\{(ih, jk), i, j = 0, \pm 1, \pm 2, \cdots\}$. Note that this procedure can be organized so that all the coefficients are determined by inverting a block triangular matrix (compare [7]) instead of applying one-dimensional interpolation at each stage.

We now turn to the higher-dimensional problem and apply the same techniques. To develop efficient computational techniques for finding interpolation points and computing the interpolating polynomial, $p_l(x)$, in n dimensions, one must first develop methods for treating this problem in $n - 1$ dimensions. For example, in the case of three dimensions, choose the origin and additional interpolation points in the $x_1 - x_2$ plane, the $x_1 - x_3$ plane and the $x_2 - x_3$ plane so that the respective polynomials $p_l(x_1, x_2, 0)$, $p_l(x_1, 0, x_3)$, $p_l(0, x_2, x_3)$ may be determined. Those are two-dimensional problems which may in turn be reduced to one-dimensional problems as we have seen. Then form

$$p_l(x_1, x_2, x_3) = p_l(x_1, x_2, 0) + p_l(x_1, 0, x_3) + p_l(0, x_2, x_3) - 2a_{00} + x_1x_2x_3q_{l-3}(x_1, x_2, x_3),$$

where $q_{l-3}(x_1, x_2, x_3)$ is a polynomial of degree $l - 3$. The discussion from this point on is analogous to that given in the two-dimensional case. It is also clear how to proceed in the n-dimensional case. Thus, even in the n-dimensional case, by choosing interpolation points appropriately on lines, we may ultimately reduce the problem to a set of one-dimensional problems.

5. Concluding Remarks. Using the idea outlined above, it is possible to obtain interpolation points for other types of polynomial interpolation problems, e.g. Hermite interpolation. The discussion parallels that already given for polynomials of degree l.

There is another type of polynomial approximation problem, however, which is of interest and which we illustrate briefly by means of an example. Consider a regular lattice in two dimensions with periods h and k. Interpolation points for the polynomial

$$p(x, y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2$$

are, e.g. $(0, 0)$, $(h, 0)$, $(-h, 0)$, $(0, k)$, $(0, -k)$ plus any one of the additional points (h, k), $(-h, k)$, $(h, -k)$, $(-h, -k)$. No matter which of the points we choose, we are led to an asymmetric distribution of the interpolation points. In many applications, however, it is desirable that the interpolation points be symmetrically distributed. If we give up the requirement that $p(x, y)$ actually takes on specified values at the four points (h, k), $(-h, k)$, $(h, -k)$, $(-h, -k)$ and simply require that $p(x, y)$ approximates a function $f(x, y)$ in some reasonable sense, let us say for example in the least squares sense, we obtain the formula

$$\begin{aligned} p(x, y) = f(0, 0) &+ \frac{f(h, 0) - f(-h, 0)}{2h} x + \frac{f(0, k) - f(0, -k)}{2k} y \\ &+ \frac{f(h, 0) - 2f(0, 0) + f(-h, 0)}{2h^2} x^2 \\ &+ \frac{f(h, k) - f(-h, k) - f(h, -k) + f(-h, -k)}{4hk} xy \\ &+ \frac{f(0, k) - 2f(0, 0) + f(0, -k)}{2k^2} y^2. \end{aligned}$$

The values of $p(x, y)$ coincide with those of $f(x, y)$ at the points $(0, 0)$, $(h, 0)$, $(-h, 0)$, $(0, k)$, $(0, -k)$, $p(x, y)$ approximates $f(x, y)$ in the least squares sense at (h, k), $(-h, k)$, $(h, -k)$, $(-h, -k)$, and the "approximation" points are symmetrically distributed, which leads to a symmetric formula for $p(x, y)$.

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