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Lower Bounds for Relatively Prime Amicable Numbers of Opposite Parity

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Abstract. Whether or not a pair of relatively prime amicable numbers exists is an open question. In this paper it is proved that if m and n are a pair of relatively prime amicable numbers of opposite parity then mn is greater than 10^{121} and m and n are each greater than 10^{60}.

1. Introduction. More than 1000 pairs of amicable numbers have been discovered to date (see [5] and the bibliography in [1]). Each of these pairs has a greatest common divisor which exceeds one, and the members of each pair are of the same parity. In [3] Kanold has shown that if m and n are relatively prime amicable numbers of opposite parity then $mn > 48 \cdot 10^{58}$. The present author showed in [2] that $mn > 10^{74}$. The purpose of the present paper is to establish a still better lower bound for mn. Thus, we shall prove the following

THEOREM. *If m and n are a pair of relatively prime amicable numbers of opposite parity then $mn > 10^{121}$.*

Our proof of this theorem is based on an extensive case study carried out on the CDC 6400 at the Temple University Computing Center. The results of a similar study involving relatively prime *odd* amicable numbers may be found in [1].

2. Some Groundwork. In this paper p and q will always represent primes while P_j will be used to denote the jth odd prime. Thus, $P_1 = 3$ and $P_{54} = 257$. If $p^a \mid mn$ but $p^{a+1} \nmid mn$ we shall write $a = \text{EXP}(p)$. m and n will be understood to be a pair of relatively prime amicable numbers of opposite parity so that

$$(1) \qquad m + n = \sigma(m) = \sigma(n),$$

where $\sigma(k)$ represents the sum of the positive divisors of k.

The following three propositions concerning mn will be needed in the next section. Although they are not new we include their proofs for completeness.

PROPOSITION 1. *If $pq \mid mn$ and* EXP $(p) = a$, *then $q \nmid \sigma(p^a)$.*

Proof. If we assume that mn has T distinct prime factors, so labeled that $p_i \mid m$ if $1 \leq i \leq s$ and $p_i \mid n$ otherwise, then from (1) and the multiplicative property of $\sigma(k)$ we have

$$(2) \qquad m + n = \prod_{i=1}^{s} \sigma(p_i^{a_i}) = \prod_{i=s+1}^{T} \sigma(p_i^{a_i}).$$

If $q \mid mn$ and $q \mid \sigma(p^a)$ we see immediately that $q \mid m$ and $q \mid n$. This is impossible since $(m, n) = 1$.

For the proof of the next proposition we require two lemmas. The first is proved on page 34 of [4]; the second follows from Theorem 22 on page 37 of [4].

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LEMMA 1. *If* $k \mid (A^2 + B^2)$ *where* $k \geqq 2$ *and* $(A, B) = 1$, *then integers* u *and* v *exist such that* $k = u^2 + v^2$.

LEMMA 2. *If* $k \mid (A^2 + 2B^2)$ *where* $k \geqq 3$ *and* $(A, 2B) = 1$, *then integers* u *and* v *exist such that* $k = u^2 + 2v^2$.

PROPOSITION 2. $mn = 2K^2$ *where* $(6, K) = 1$.

Proof. Using the notation introduced in the proof of Proposition 1 let $p_1 = 2$ and $a_1 = t$. Then from (2) we have

$$m + n = (2^{t+1} - 1) \prod_{i=2}^{s} \sigma(p_i^{a_i}) = \prod_{i=s+1}^{T} \sigma(p_i^{a_i}). \tag{3}$$

Since $m + n$ is odd, and since for an odd prime $\sigma(p^a)$ is odd if and only if a is even, we see that a_i is even for $2 \leqq i \leqq T$. Therefore, $mn = 2^t K^2$ where K is odd.

Now assume that t is even so that $m + n$ is the sum of two relatively prime squares. $2^{t+1} - 1 \equiv 3 \pmod 4$ so that $2^{t+1} - 1$ has a prime factor P of the form $4k + 3$. Since, from (3), $P \mid (m + n)$ it follows from Lemma 1 that P is the sum of two squares. But this is impossible since $P \not\equiv 1 \pmod 4$, and we conclude that t is odd.

If t is odd and $t > 1$, then $2^{t+1} - 1 \equiv -1 \pmod 8$ so that $2^{t+1} - 1$ is divisible by a prime Q such that $Q = 8k + 5$ or $Q = 8k + 7$. Also, $m + n = 2B^2 + A^2$ where $(A, 2B) = 1$, and since $Q \mid (m + n)$ it follows from Lemma 2 that $Q = u^2 + 2v^2$. Since $u^2 + 2v^2 \not\equiv 5, 7 \pmod 8$ we have a contradiction. Therefore, $t = 1$ and $mn = 2K^2$.

Since $t = 1$ we see from (3) that $3 \mid (m + n)$. Therefore, if $3 \mid mn$ then $3 \mid m$ and $3 \mid n$ which is impossible since $(m, n) = 1$. Thus, $3 \nmid mn$ and the proof is complete.

PROPOSITION 3. *If* $p \mid mn$ *and* EXP $(p) = a$ *then* (i) *if* $p = 8k + 1$ *then* $a \equiv 0, 2 \pmod 8$; (ii) *if* $p = 8k + 3$ *then* $4 \mid a$; (iii) *if* $p = 8k + 5$ *then* $a \equiv 0, 6 \pmod 8$; (iv) *if* $p = 8k + 7$ *then* $2 \mid a$.

Proof. From Proposition 2 we know already that a is even so that there is nothing to prove in case (iv). From Proposition 2, Lemma 2, (2) and the fact that if u is odd then $u^2 + 2v^2 \equiv 1, 3 \pmod 8$, we see that $\sigma(p^a) \equiv 1, 3 \pmod 8$.

If $p = 8k + 1$ then $\sigma(p^a) \equiv 1 + p + \cdots + p^a \equiv 1 + a \pmod 8$. Therefore, $1 + a \equiv 1, 3 \pmod 8$ and (i) follows.

If $p = 8k + 3$ then $\sigma(p^a) \equiv 1 + 3 + 1 + \cdots + 3 + 1 \equiv 1 + 2a \pmod 8$. Therefore, $1 + 2a \equiv 1, 3 \pmod 8$ and (ii) follows.

If $p = 8k + 5$ then $\sigma(p^a) \equiv 1 + 5 + 1 + \cdots + 5 + 1 \equiv 1 + 3a \pmod 8$. Therefore, $1 + 3a \equiv 1, 3 \pmod 8$ and (iii) follows.

3. A Lower Bound for *mn*.

In proving our theorem we shall consider 27 mutually exclusive and exhaustive cases which are distinguished by our knowledge as to whether each prime in a subset selected from the set $S = \{5, 7, 11, 13, 19, 23, 29, 37, 43, 47, 53, 59\}$ does, or does not, divide mn. Our findings appear in Table II in which the presence of a + in the column headed by p indicates that $p \mid mn$ while the presence of a 0 indicates that $p \nmid mn$.

Using (1), Proposition 2, the multiplicative property of $\sigma(k)$, and the fact that $\sigma(p^a)/p^a < p/(p - 1)$ we see that

$$4 < 2 + m/n + n/m = \sigma(mn)/mn = 1.5 \prod \sigma(p^a)/p^a < 1.5 \prod p/(p - 1) \tag{4}$$

where the products are taken over the odd prime divisors of mn and $a =$ EXP (p). If mn has T distinct prime factors, and if it is known that mn is not divisible by any

TABLE I. *Pertinent Prime Divisors of* $\sigma(p^a)$

p \ a	2	4	6	8	10	12	14	16
5	X	X	NONE		X	X		
7	19	NONE						
11	X	5	X	7,19	X	NONE	X	
13	X	X	NONE		X	X		
19	X	NONE	X		X		X	
23	7	NONE						
29	X	X	7	13	X	X	13	NONE
37	X	X	NONE		X	X		
43	X	NONE	X		X		X	
47	37	11	43	19,37	NONE			
53	X	X	29	7,37	X	X	7,11	NONE
59	X	11	X	NONE	X		X	

TABLE II

Divisibility Restrictions on mn.												N	Lower Bound For mn
5	7	11	13	19	23	29	37	43	47	53	59		
0												53	$2\cdot 7^2 11^4 19^4\cdot Q_{49}(593) > 10^{238}$
+	0											39	$2\cdot 5^6 19^4\cdot Q_{36}(409) > 10^{166}$
+	+	0										30	$2\cdot 5^6 7^2\cdot Q_{27}(433) > 10^{124}$
+	+	+	0									28	$2\cdot 5^6 7^2 11^{12}\cdot Q_{24}(367) > 10^{121}$
+	+	+	+	0		+						25	$2\cdot 5^6 7^2 11^{12} 13^6 29^{16}\cdot Q_{19}(311) > 10^{127}$
+	+	+	+	0		0						28	$2\cdot 5^6 7^2 11^{12} 13^6\cdot Q_{23}(383) > 10^{124}$
+	+	+	+	+	0	+						24	$2\cdot 5^6 7^4 11^{12} 13^6 19^4 29^{16}\cdot Q_{17}(281) > 10^{124}$
+	+	+	+	+	0	0						27	$2\cdot 5^6 7^4 11^{12} 13^6 19^4\cdot Q_{21}(409) > 10^{121}$
+	+	+	+	+	+	0	0	+		+	+	25	$2\cdot 5^6 7^4 11^{12} 13^6 19^4 23^4 43^4 53^6 59^8\cdot Q_{15}(257) > 10^{127}$
+	+	+	+	+	+	0	0	+		+	0	26	$2\cdot 5^6 7^4 11^{12} 13^6 19^4 23^4 43^4 53^6\cdot Q_{17}(281) > 10^{123}$
+	+	+	+	+	+	0	0	+		0	+	26	$2\cdot 5^6 7^4 11^{12} 13^6 19^4 23^4 43^4 59^8\cdot Q_{17}(281) > 10^{126}$
+	+	+	+	+	+	0	0	+		0	0	27	$2\cdot 5^6 7^4 11^{12} 13^6 19^4 23^4 43^4\cdot Q_{19}(383) > 10^{122}$
+	+	+	+	+	+	0	0	0				27	$2\cdot 5^6 7^4 11^{12} 13^6 19^4 23^4\cdot Q_{20}(383) > 10^{121}$

member of a subset of r given primes taken from S, then from (4) and the monotonic decreasing nature of the function $x/(x-1)$ it follows that $4 < \prod^* P_j/(P_j - 1)$, where $1 \leqq j \leqq T + r$ and the asterisk indicates the omission of each of the r specified primes. (Note that $1.5 = P_1/(P_1 - 1)$ and recall from Proposition 2 that $3 \nmid mn$.) We see immediately that a lower bound for T, denoted by N in Table II, can be determined by finding the smallest integer M such that

$$4 < \prod_{j=1}^{M}{}^{*} P_j/(P_j - 1).$$

Armed with this lower bound for the number of prime divisors of mn, it is then possible to establish lower bounds for mn in each case. Here the use of Propositions 1, 2, 3 is essential and, in particular, a study of the divisibility of $\sigma(p^a)$ by q, where both p and q belong to S, and where a is restricted in accordance with the conclusions of Proposition 3, is necessary. Due to the magnitude of the numbers involved as well as the multiplicity of cases the investigation was carried out on the CDC 6400

TABLE II (*Continued*)

5	7	11	13	19	23	29	37	43	47	53	59	N	Lower Bound For mn
+	+	+	+	+	+	0	+	+	+	+		23	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 37^6 43^4 47^{10} 53^6 \cdot Q_{12}(223) > 10^{126}$
										0		24	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 37^6 43^4 47^{10} \cdot Q_{14}(257) > 10^{125}$
									0	+	+	24	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 37^6 43^4 53^6 59^8 \cdot Q_{13}(241) > 10^{128}$
										+	0	25	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 37^6 43^4 53^6 \cdot Q_{15}(271) > 10^{124}$
										0	+	25	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 37^6 43^4 59^8 \cdot Q_{15}(271) > 10^{128}$
										0	0	26	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 37^6 43^4 \cdot Q_{17}(313) > 10^{123}$
								0	+			25	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 37^6 47^6 \cdot Q_{16}(281) > 10^{122}$
									0			26	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 37^6 \cdot Q_{18}(337) > 10^{122}$
+	+	+	+	+	+	+	0	+		+		22	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 29^{16} 43^4 53^{16} \cdot Q_{12}(199) > 10^{139}$
										0		23	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 29^{16} 43^4 \cdot Q_{14}(241) > 10^{121}$
								0				24	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 29^{16} \cdot Q_{16}(271) > 10^{124}$
+	+	+	+	+	+	+	+	+			+	21	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 29^{16} 37^6 43^4 59^8 \cdot Q_{10}(167) > 10^{127}$
											0	22	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 29^{16} 37^6 43^4 \cdot Q_{12}(223) > 10^{122}$
								0				22	$2\cdot5^6 7^4 11^{12} 13^6 19^4 23^4 29^{16} 37^6 \cdot Q_{13}(241) > 10^{121}$

using modular arithmetic. The *pertinent* prime divisors of $\sigma(p^a)$ are given in Table I. An entry of X in this table indicates that, in accordance with Proposition 3, $a =$ EXP (p) is impossible. If $\sigma(p^a)$ has no pertinent divisors and $b > a$ then the divisors of $\sigma(p^b)$ are not tabulated.

From this table we see, for example, that EXP (43) $\geqq$ 4 while if $7 \mid mn$ then EXP (7) $\geqq$ 4 or EXP (7) $\geqq$ 2 according as 19 does or does not divide mn. If $7, 53 \mid mn$ or $11, 37, 53 \mid mn$ then EXP (53) $\geqq$ 16 or EXP (53) $\geqq$ 6 according as 29 does or does not divide mn.

One last word of explanation concerning Table II is in order. In each case $Q_k(q)$ denotes the product of the squares of the k primes between 17 and q, inclusive, which are congruent to 1 or 7 modulo 8 and which satisfy the condition imposed by Proposition 1. That is, if P is one of these primes then $\sigma(P^2)$ is not divisible by any prime known to be a divisor of mn. For example, in Case 25

$$Q_{10}(167) = (17\cdot31\cdot41\cdot71\cdot73\cdot89\cdot97\cdot103\cdot127\cdot167)^2.$$

For each P in this product $p \nmid \sigma(P^2)$, where $p =$ 5, 7, 11, 13, 19, 23, 29, 37, 43, 59.

4. Lower Bounds for m and n. We may, without loss of generality, assume that m is even. Then, according to Corollary 1.3 of [2], $m < 2n$. Employing our theorem we have $2n^2 > mn > 10^{121}$, so that $n > 10^{60}$. If $m > n$ then $m > 10^{60}$ also. If $m < n$ there are two possibilities. If $4m > n$ then $4m^2 > mn > 10^{121}$ and $m > 10^{60}$. If $4m < n$ then from (1) and considerations similar to those of Section 3 we have

$$5 < (m + n)/m = \sigma(m)/m < \prod_{i=1}^{R} P_i/(P_i - 1),$$

where R is the number of primes which divide m. Therefore, if M is the smallest integer such that

$$5 < \prod_{i=1}^{M} P_i/(P_i - 1),$$

then certainly $m > 2(5 \cdot 7 \cdots P_M)^2$. It was found that $M = 54$ and $m > 10^{205}$. We have proved the following

COROLLARY. *If m and n are relatively prime amicable numbers of opposite parity then $m > 10^{60}$ and $n > 10^{60}$.*

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