MATHEMATICS OF COMPUTATION, VOLUME 25, NUMBER 113, JANUARY, 1971

An Implemention of Christoffel's Theorem in the Theory of Orthogonal Polynomials

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Abstract. An algorithm for the construction of the polynomials associated with the weight function $w(t)P(t)$ from those associated with $w(t)$ is given for the case when $P(t)$ is a polynomial which is nonnegative in the interval of orthogonality. The relation of the algorithm to the LR algorithm is also discussed.

Introduction. In several problems of numerical analysis, particularly in the construction of Gaussian quadrature rules with preassigned nodes, the following problem arises. Given the orthogonal polynomials $\{p_j(t)\}$ associated with a weight function $w(t)$ on the interval (a, b) and a polynomial $P(t)$ of degree m which is nonnegative on the interval (a, b), construct the orthogonal polynomials $\{q_j(t)\}$ associated with the weight function $P(t)w(t)$ on the same interval.

A theorem of Christoffel [1] gives an explicit expression for the polynomial $q_n(t)$ in the form

$$q_n(t)P(t) = \begin{vmatrix} p_n(t) & p_{n+1}(t) & \cdots & p_{n+m}(t) \\ p_n(z_1) & p_{n+1}(z_1) & \cdots & p_{n+m}(z_1) \\ p_n(z_2) & p_{n+1}(z_2) & \cdots & p_{n+m}(z_2) \\ \vdots & \vdots & & \vdots \\ p_n(z_m) & p_{n+1}(z_m) & \cdots & p_{n+m}(z_m) \end{vmatrix}, \tag{1}$$

where the z_k, $k = 1(1)m$, are the roots of $P(t)$. If some root, z_i, is of multiplicity j, then the corresponding rows of (1) are replaced by the derivatives of order $0, 1, \cdots, j - 1$ of the polynomials $p_r(t)$, $r = n(1)n + m$, at $t = z_i$. For numerical calculations, Eq. (1) is very clumsy to use, even for simple evaluation of the polynomial $q_n(t)$ at a point, unless m is small. Often, the three-term recurrence relation

$$p_j(t) = (t - b_j)p_{j-1}(t) - g_jp_{j-2}(t), \qquad j = 1, 2, \cdots, \tag{2}$$

with $p_0(t) = 1$ and $p_{-1}(t) = 0$, is known because it is more convenient to obtain [4], [5] and to use [5], [6]. The main result of this paper is to prove a theorem, equivalent to Christoffel's, which states an explicit construction of the three-term recurrence relation

$$q_j(t) = (t - B_j)q_{j-1}(t) - G_jq_{j-2}(t), \qquad j = 1, 2, \cdots, \tag{3}$$

Received March 9, 1970, revised May 13, 1970.

AMS 1969 subject classifications. Primary 6510; Secondary 6525, 6550, 3345.

Key words and phrases. Numerical construction of orthogonal polynomials, quotient-difference algorithm, LR algorithm, Gaussian quadrature, three-term recurrence relations for orthogonal polynomials.

with $q_0(t) = 1$ and $q_{-1}(t) = 0$, from (2), at least when $P(t)$ has no roots in the interval of orthogonality. A link with the LR algorithm [3] is shown so that the theory of the latter can be used to establish the stability and also to modify the algorithm for constructing (3).

Main Result. Let $P(t)$ be a polynomial of degree m which is strictly positive on (a, b) with roots $z_1, z_2, \cdots, z_m$. Define $B_j^{(0)} = b_j$, $G_j^{(0)} = g_j$, $j = 1, 2, \cdots$, and further define $B_j^{(k)}$, $G_j^{(k)}$, $j = 1, 2, \cdots; k = 1, 2, \cdots, m$, by

$$B_j^{(k)} = z_k + Q_j + E_j, \qquad G_j^{(k)} = Q_j E_{j-1}, \tag{4}$$

where

$$\begin{aligned} E_0 &= 0, \\ Q_j &= B_j^{(k-1)} - E_{j-1} - z_k \qquad (j = 1, 2, \cdots), \\ E_j &= G_{j+1}^{(k-1)}/Q_j. \end{aligned} \tag{5}$$

Then the parameters of (3) are given by $B_j = B_j^{(m)}$, $G_j = G_j^{(m)}$, for $j = 1, 2, \cdots$.

The assertion for general m follows from the result for $m = 1$. For $m = 1$, the result can be established easily. If (a, b) does not contain the origin and $P(t) = t$, then the quotient-difference algorithm implies [2] that $B_j^{(1)}$, $G_j^{(1)}$, $j = 1, 2, \cdots$, are the parameters of the three-term recurrence relation for the monic orthogonal polynomials associated with $tw(t)$ on (a, b). For $P(t) = (t - z_1)$, steps (5) and (4) consist of the following sequence of rules:

(a) perform the transformation of variables $u = t - z_1$;

(b) apply the quotient-difference algorithm step as before;

(c) perform the transformation of variables $t = u + z_1$.

It is easy to verify that $B_j^{(1)}$, $G_j^{(1)}$, $j = 1, 2, \cdots$, are the parameters of the three-term recurrence relation for the monic orthogonal polynomials associated with $(t - z_1)w(t)$ on the interval (a, b). The result here does not depend upon z_1 being real, only that z_1 is not interior to (a, b). In this case, orthogonality means $\int_a^b (t - z_1)w(t)r_j(t)r_k(t)dt = 0$ when $j \neq k$, where $r_0(t) = 1$, $r_{-1}(t) = 0$, and $r_j(t) = [t - B_j^{(1)}]r_{j-1}(t) - G_j^{(1)}r_{j-2}(t)$ for $j = 1, 2, \cdots$.

Discussion. In practice, only a finite number, say n, of the parameters B_j, G_j are desired. It is clear from the construction (5) and (4) that when $P(t)$ is a polynomial of degree m, then $n + m$ of the b_j, g_j are required. The rules are then modified to

$$\begin{aligned} E_0 &= 0, \\ Q_j &= B_j^{(k-1)} - E_{j-1} - z_k, \\ E_j &= G_{j+1}^{(k-1)}/Q_j \qquad (j = 1, 2, \cdots, n + m - k;\ k = 1, 2, \cdots, m), \\ B_j^{(k)} &= Q_j + E_j + z_k, \\ G_j^{(k)} &= Q_j E_{j-1}. \end{aligned} \tag{6}$$

These rules may be interpreted in terms of matrix decompositions. Let

$$A_k = \begin{bmatrix} B_1^{(k)} & 1 & & & & \bigcirc \\ G_2^{(k)} & B_2^{(k)} & 1 & & & \\ & G_3^{(k)} & B_3^{(k)} & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ \bigcirc & & & & G_{n+m-k}^{(k)} & B_{n+m-k}^{(k)} \end{bmatrix}, \quad k = 0(1)m-1.$$

Then one step of (6) may be interpreted as follows: write $LR = A_k - z_{k+1}I$, $C_k = RL + z_{k+1}I$, discard the last row and column of C_k and the result is A_{k+1}. The matrix L is lower triangular with unit diagonal, R is upper triangular, and I is the identity matrix. The formation of C_k from A_k is one step of the LR algorithm [3] without interchanges and with origin shift z_{k+1}. This decomposition exists whenever z_{k+1} is not an eigenvalue of any of the principal minors of A_k. Therefore, the stability and existence of the construction (6) are identical to those of the LR algorithm without interchanges. At least for the important case when the z_k are all real and outside the interval (a, b), the construction (6) is numerically quite stable. The identification of (6) with the LR algorithm also yields a method for avoiding complex arithmetic by the use of the double step LR process for pairs of complex conjugate roots of $P(t)$.

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