An Implementation of Christoffel's Theorem in the Theory of Orthogonal Polynomials

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Abstract. An algorithm for the construction of the polynomials associated with the weight function \( w(t)P(t) \) from those associated with \( w(t) \) is given for the case when \( P(t) \) is a polynomial which is nonnegative in the interval of orthogonality. The relation of the algorithm to the LR algorithm is also discussed.

Introduction. In several problems of numerical analysis, particularly in the construction of Gaussian quadrature rules with preassigned nodes, the following problem arises. Given the orthogonal polynomials \( \{ p_i(t) \} \) associated with a weight function \( w(t) \) on the interval \((a, b)\) and a polynomial \( P(t) \) of degree \( m \) which is nonnegative on the interval \((a, b)\), construct the orthogonal polynomials \( \{ q_i(t) \} \) associated with the weight function \( P(t)w(t) \) on the same interval.

A theorem of Christoffel [1] gives an explicit expression for the polynomial \( q_n(t) \) in the form

\[
q_n(t) = \frac{P_0(z_n)P_{n+1}(z_n)\cdots P_{n+m}(z_n)}{P_{n}(z_n)P_{n+1}(z_n)\cdots P_{n+m}(z_n)}
\]

where the \( z_k, k = 1(1)m \), are the roots of \( P(t) \). If some root, \( z_k \), is of multiplicity \( j \), then the corresponding rows of (1) are replaced by the derivatives of order 0, 1, \( \cdots \), \( j-1 \) of the polynomials \( p_i(t) \), \( r = n(1)n + m \), at \( t = z_k \). For numerical calculations, Eq. (1) is very clumsy to use, even for simple evaluation of the polynomial \( q_n(t) \) at a point, unless \( m \) is small. Often, the three-term recurrence relation

\[
p_j(t) = (t - b_j)p_{j-1}(t) - g_jp_{j-2}(t), \quad j = 1, 2, \cdots
\]

with \( p_0(t) = 1 \) and \( p_{-1}(t) = 0 \), is known because it is more convenient to obtain [4], [5] and to use [5], [6]. The main result of this paper is to prove a theorem, equivalent to Christoffel’s, which states an explicit construction of the three-term recurrence relation

\[
q_i(t) = (t - B_i)q_{i-1}(t) - G_iq_{i-2}(t), \quad j = 1, 2, \cdots
\]
with \( q_0(t) = 1 \) and \( q_{-1}(t) = 0 \), from (2), at least when \( P(t) \) has no roots in the interval of orthogonality. A link with the LR algorithm [3] is shown so that the theory of the latter can be used to establish the stability and also to modify the algorithm for constructing (3).

**Main Result.** Let \( P(t) \) be a polynomial of degree \( m \) which is strictly positive on \((a, b)\) with roots \( z_1, z_2, \cdots, z_m\). Define \( B_i^{(0)} = b_i, G_i^{(0)} = g_i, j = 1, 2, \cdots, \) and further define \( B_i^{(k)}, G_i^{(k)}, j = 1, 2, \cdots; k = 1, 2, \cdots, m, \) by

\[
B_i^{(k)} = z_k + Q_i + E_k, \quad G_i^{(k)} = Q_i E_{i-1},
\]

where

\[
Q_i = B_i^{(k-1)} - E_{i-1} - z_k \quad (j = 1, 2, \cdots),
\]

\[
E_i = G_{i+1}^{(k-1)} / Q_i.
\]

Then the parameters of (3) are given by \( B_i = B_i^{(m)}, G_i = G_i^{(m)} \), for \( j = 1, 2, \cdots \).

The assertion for general \( m \) follows from the result for \( m = 1 \). For \( m = 1 \), the result can be established easily. If \((a, b)\) does not contain the origin and \( P(t) = t \), then the quotient-difference algorithm implies [2] that \( B_j^{(1)}, G_j^{(1)}, j = 1, 2, \cdots, \) are the parameters of the three-term recurrence relation for the monic orthogonal polynomials associated with \( t w(t) \) on \((a, b)\). For \( P(t) = (t - z) \), steps (5) and (4) consist of the following sequence of rules:

(a) perform the transformation of variables \( u = t - z \);

(b) apply the quotient-difference algorithm step as before;

(c) perform the transformation of variables \( t = u + z \).

It is easy to verify that \( B_j^{(1)}, G_j^{(1)}, j = 1, 2, \cdots, \) are the parameters of the three-term recurrence relation for the monic orthogonal polynomials associated with \((t - z)w(t)\) on the interval \((a, b)\). The result here does not depend upon \( z \) being real, only that \( z \) is not interior to \((a, b)\). In this case, orthogonality means \( \int_a^b (t - z)w(t)r_j(t)r_k(t)dt = 0 \) when \( j \neq k \), where \( r_0(t) = 1, r_{-1}(t) = 0 \), and \( r_j(t) = [t - B_j^{(1)}]r_{j-1}(t) - G_j^{(1)}r_{j-2}(t) \) for \( j = 1, 2, \cdots \).

**Discussion.** In practice, only a finite number, say \( n \), of the parameters \( B_i, G_i \) are desired. It is clear from the construction (5) and (4) that when \( P(t) \) is a polynomial of degree \( m \), then \( n + m \) of the \( b_i, g_i \) are required. The rules are then modified to

\[
E_0 = 0,
\]

\[
Q_i = B_i^{(k-1)} - E_{i-1} - z_k,
\]

\[
E_i = G_{i+1}^{(k-1)} / Q_i \quad (j = 1, 2, \cdots, n + m - k; k = 1, 2, \cdots, m),
\]

\[
B_i^{(k)} = Q_i + E_i + z_k,
\]

\[
G_i^{(k)} = Q_i E_{i-1}.
\]

These rules may be interpreted in terms of matrix decompositions. Let
Then one step of (6) may be interpreted as follows: write $LR = A_k - z_{k+1}I$, $C_k = RL + z_{k+1}I$, discard the last row and column of $C_k$ and the result is $A_{k+1}$. The matrix $L$ is lower triangular with unit diagonal, $R$ is upper triangular, and $I$ is the identity matrix. The formation of $C_k$ from $A_k$ is one step of the $LR$ algorithm [3] without interchanges and with origin shift $z_{k+1}$. This decomposition exists whenever $z_{k+1}$ is not an eigenvalue of any of the principal minors of $A_k$. Therefore, the stability and existence of the construction (6) are identical to those of the $LR$ algorithm without interchanges. At least for the important case when the $z_k$ are all real and outside the interval $(a, b)$, the construction (6) is numerically quite stable. The identification of (6) with the $LR$ algorithm also yields a method for avoiding complex arithmetic by the use of the double step $LR$ process for pairs of complex conjugate roots of $P(t)$.