Uniform Asymptotic Expansions of the Jacobi Polynomials and an Associated Function

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Abstract. Asymptotic expansions have been obtained using two theorems due to Olver for the Jacobi polynomials and an associated function. These expansions are uniformly valid for complex arguments over certain regions, for large values of the order.

1. Introduction. In order to make realistic estimates of the truncation errors in Gauss-Jacobi quadrature rules (see Donaldson and Elliott [1]), we have needed asymptotic estimates for large n and $z \notin [-1, 1]$ of the Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ and an associated function which we have denoted by $\Pi_n^{(\alpha,\beta)}(z)$, and which is defined by

$$(1.1) \qquad \Pi_n^{(\alpha,\beta)}(z) = \int_{-1}^{1} \frac{(1-t)^\alpha (1+t)^\beta P_n^{(\alpha,\beta)}(t)}{z-t} \, dt, \qquad \text{for} \quad z \notin [-1, 1].$$

The function $\Pi_n^{(\alpha,\beta)}(z)$ is analytic in the complex plane cut along $[-1, 1]$ and is closely related to the Jacobi function of the second kind, $Q_n^{(\alpha,\beta)}(z)$ as defined by Szegö [7]. In terms of the hypergeometric function we have,

$$
\begin{aligned}
(1.2) \qquad \Pi_n^{(\alpha,\beta)}(z) = {} & \frac{2^{n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)(z-1)^{n+1}} \\
& \cdot {}_2F_1\!\left(n+1, n+\alpha+1; 2n+\alpha+\beta+2; \frac{2}{1-z}\right).
\end{aligned}
$$

The problem of finding such asymptotic estimates for large n and z, not in the neighborhood of $[-1, 1]$, has already received some attention in the literature. Erdélyi [2, pp. 77–78] quotes results given by Watson for the hypergeometric function, which will give the first term in the asymptotic expansions of $P_n^{(\alpha,\beta)}(z)$ and $\Pi_n^{(\alpha,\beta)}(z)$. Szegö [7, Theorem 8.21.7] also gives the first term for $P_n^{(\alpha,\beta)}(z)$ and, in Eq. (8.71.19) of [7], gives the form of the first term of the asymptotic expansion of $\Pi_n^{(\alpha,\beta)}(z)$.

In this paper, we have made use of two of the theorems developed by Olver [5], [6] to obtain formal asymptotic *expansions* of both $P_n^{(\alpha,\beta)}(z)$ and $\Pi_n^{(\alpha,\beta)}(z)$ for large n. In Section 2, we shall obtain these expansions in terms of elementary functions which are valid in the complex z-plane cut along $[-1, 1]$, with a neighborhood of the interval $[-1, 1]$ being deleted. In Section 3, we shall derive expansions in terms of modified Bessel functions which again are valid in the cut plane, but also in the neighborhood of the point $z = 1$; in fact, the expansions are valid in the complex plane cut along $[-1, 1]$ but with only a neighborhood of the point $z = -1$ being deleted.

The results of this section are believed to be new. Finally, in Section 4, we shall give explicit relations between the coefficients which arise in the asymptotic expansions of Sections 2 and 3.

In this paper, we shall obtain only the formal asymptotic expansions. It is proposed to consider in a later paper the error bounds when the asymptotic expansions are truncated.

2. Asymptotic Expansions when z is not in the Neighborhood of $[-1, 1]$.

The starting point of all the subsequent analysis is the observation that the functions $P_n^{(\alpha,\beta)}(z)$ and $(z-1)^{-\alpha}(z+1)^{-\beta}\Pi_n^{(\alpha,\beta)}(z)$, where $-\pi < \arg(z-1)$, $\arg(z+1) < \pi$, are linearly independent solutions of the differential equation

$$(2.1) \quad (z^2 - 1)\frac{d^2\theta}{dz^2} + [(\alpha + \beta + 2)z + (\alpha - \beta)]\frac{d\theta}{dz} - n(n + \alpha + \beta + 1)\theta = 0.$$

We first reduce this equation to "normal form" by writing

$$(2.2) \qquad\qquad \theta(z) = (z - 1)^{-(\alpha+1)/2}(z + 1)^{-(\beta+1)/2}u(z),$$

which gives

$$(2.3) \qquad \frac{d^2u}{dz^2} = \left\{ \frac{k^2}{(z^2 - 1)} + \frac{\alpha^2 - 1}{4(z - 1)^2} + \frac{1 - \alpha^2 - \beta^2}{4(z^2 - 1)} + \frac{\beta^2 - 1}{4(z + 1)^2} \right\}u.$$

The quantity k is defined by

$$(2.4) \qquad\qquad\qquad k = n + (\alpha + \beta + 1)/2.$$

We are interested in the asymptotic expansions of the solutions of this equation for large values of k. Following Olver [5], we make a simultaneous transformation of both the dependent and independent variables. If we define

$$(2.5) \qquad\qquad\qquad z = \cosh 2\zeta,$$

and

$$(2.6) \qquad\qquad\qquad u = (\sinh 2\zeta)^{1/2}w(\zeta),$$

Eq. (2.3) becomes

$$(2.7) \qquad\qquad\qquad \frac{d^2w}{d\zeta^2} = \{4k^2 + f(\zeta)\}w,$$

where the function $f(\zeta)$ is defined by

$$(2.8) \qquad f(\zeta) = \frac{2\alpha^2}{\cosh 2\zeta - 1} - \frac{2\beta^2}{\cosh 2\zeta + 1} - \frac{1}{(\sinh 2\zeta)^2}.$$

We can now apply Olver's Theorem A (see [5]) to this equation. We note that Eq. (2.5) maps the z-plane cut along $[-1, 1]$ into the domain D in the ζ-plane defined by

$$(2.9) \qquad\qquad D \equiv \{\zeta \mid \operatorname{Re}\zeta > 0,\ -\pi/2 < \operatorname{Im}\zeta < \pi/2\}.$$

If we delete the neighborhoods of the points $\zeta = 0$, $\pm i\pi/2$, we have immediately from Olver's Theorem A, that in the domain D Eq. (2.7) possesses two linearly in-

dependent solutions $w_1(\zeta)$ and $w_2(\zeta)$, say, which for large k are represented asymptotically by

$$(2.10) \qquad w_1(\zeta) \sim e^{2k\zeta} \sum_{s=0}^{\infty} \frac{a_s(\zeta)}{(2k)^s},$$

and

$$(2.11) \qquad w_2(\zeta) \sim e^{-2k\zeta} \sum_{s=0}^{\infty} \frac{(-1)^s a_s(\zeta)}{(2k)^s}.$$

The coefficients $a_s(\zeta)$ are defined recursively by

$$a_0(\zeta) = 1,$$

$$(2.12) \qquad a_{s+1}(\zeta) = -\tfrac{1}{2} a_s'(\zeta) - \tfrac{1}{2} \int_{\zeta}^{\infty} f(t) a_s(t)\, dt,$$

for $s = 0, 1, 2, \cdots$, where the constant of integration has been chosen so that $a_{s+1}(\infty)$ $= 0$ for $s = 0, 1, 2, \cdots$. It now remains to relate the functions $w_1(\zeta)$ and $w_2(\zeta)$ to our original functions $P_n^{(\alpha,\beta)}(z)$ and $\Pi_n^{(\alpha,\beta)}(z)$. This is a straightforward process done by comparing these four functions, assuming k to be fixed, for large $|z|$, or large $|\zeta|$. We find, after some algebraic computation, that the required asymptotic expansions are given by

$$(2.13) \qquad \begin{aligned} P_n^{(\alpha,\beta)}(z) &\sim \frac{\Gamma(2n + \alpha + \beta + 1)}{2^{[2n+(\alpha+\beta+1)/2]}\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} \\ &\quad \times \frac{(z + (z^2 - 1)^{1/2})^k}{(z-1)^{(2\alpha+1)/4}(z+1)^{(2\beta+1)/4}} \sum_{s=0}^{\infty} \frac{a_s(\zeta)}{(2k)^s}, \end{aligned}$$

and

$$(2.14) \qquad \begin{aligned} \Pi_n^{(\alpha,\beta)}(z) &\sim \frac{2^{[2n+3(\alpha+\beta+1)/2]}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)} \\ &\quad \times \frac{(z-1)^{(2\alpha-1)/4}(z+1)^{(2\beta-1)/4}}{[z + (z^2-1)^{1/2}]^k} \sum_{s=0}^{\infty} \frac{(-1)^s a_s(\zeta)}{(2k)^s}. \end{aligned}$$

3. Asymptotic Expansions Valid Near $z = 1$.

We again start with Eq. (2.3). If we introduce the same independent variable as before (see Eq. (2.5)), but define a new dependent variable by

$$(3.1) \qquad u = \left(\frac{\sinh 2\zeta}{\zeta}\right)^{1/2} W(\zeta),$$

Eq. (2.3) becomes,

$$(3.2) \qquad \frac{d^2 W}{d\zeta^2} = \frac{1}{\zeta}\frac{dW}{d\zeta} + \left\{ 4k^2 + \frac{\alpha^2 - 1}{\zeta^2} + F(\zeta) \right\} W.$$

The function $F(\zeta)$ is defined by

$$(3.3) \qquad F(\zeta) = (\alpha^2 - 1)\left[\frac{1}{\sinh^2 \zeta} - \frac{1}{\zeta^2}\right] - \frac{\beta^2 - 1}{\cosh^2 \zeta} + 3\left[\frac{1}{\sinh^2 2\zeta} - \frac{1}{(2\zeta)^2}\right].$$

Equation (3.2) is now in a form suitable for the application of Olver's Theorem D (see [6]). The asymptotic expansions will again be valid in the domain D (see Eq. (2.9)) but only with the neighborhoods of the points $\pm i\pi/2$ deleted. We note that $F(\zeta)$ is an even and regular function of ζ such that $F(\zeta) \sim O(|\zeta|^{-2})$ as $|\zeta| \to \infty$, $\zeta \in D$. By Olver's Theorem D, for $\alpha \geqq 0$, Eq. (3.2) possesses two linearly independent solutions $W_1(\zeta)$ and $W_2(\zeta)$, say, which have asymptotic expansions

$$(3.4) \qquad W_1(\zeta) \sim \zeta I_\alpha(2k\zeta) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{(2k)^{2s}} + \zeta I_{\alpha+1}(2k\zeta) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{(2k)^{2s+1}},$$

and

$$(3.5) \qquad W_2(\zeta) \sim \zeta K_\alpha(2k\zeta) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{(2k)^{2s}} - \zeta K_{\alpha+1}(2k\zeta) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{(2k)^{2s+1}}.$$

In these equations, $I_\alpha(z)$ and $K_\alpha(z)$ denote the modified Bessel functions of order α (see for example, [3]). The functions $A_s(\zeta)$, $B_s(\zeta)$ are defined recursively by

$$A_0(\zeta) = 1,$$

$$(3.6) \qquad 2B_s(\zeta) = -A_s'(\zeta) + \int_0^\zeta \left\{ F(t)A_s(t) - \frac{2\alpha+1}{t} A_s'(t) \right\} dt,$$

$$2A_{s+1}(\zeta) = \frac{2\alpha+1}{\zeta} B_s(\zeta) - B_s'(\zeta) + \int_0^\zeta F(t)B_s(t)\, dt + k_{s+1},$$

for $s = 0, 1, 2, \cdots$. The constants k_{s+1} are chosen so that

$$(3.7) \qquad A_{s+1}(0) = 0 \quad \text{for } s = 0, 1, 2, \cdots.$$

It is worth noting that $A_s(\zeta)$, $B_s(\zeta)$ are even and odd functions of ζ respectively.

It now remains to identify the functions $W_1(\zeta)$ and $W_2(\zeta)$ with $P_n^{(\alpha,\beta)}(z)$ and $\Pi_n^{(\alpha,\beta)}(z)$. This is done by taking k fixed, and considering the behavior of these functions in the limit as $z \to 1$ or $\zeta \to 0$, $\zeta \in D$. Since

$$(3.8) \qquad W_1(\zeta) \sim \frac{k^\alpha}{\Gamma(\alpha+1)} \zeta^{\alpha+1}[1 + O(\zeta^2)],$$

and

$$(3.9) \qquad P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} [1 + O(z-1)],$$

we find that

$$(3.10) \qquad \begin{aligned} P_n^{(\alpha,\beta)}(z) \sim\ & \frac{2^{(\alpha+\beta+1)/2}\Gamma(n+\alpha+1)}{k^\alpha \Gamma(n+1)(z-1)^{(2\alpha+1)/4}(z+1)^{(2\beta+1)/4}} \\ & \times \left\{ \zeta^{1/2} I_\alpha(2k\zeta) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{(2k)^{2s}} + \zeta^{1/2} I_{\alpha+1}(2k\zeta) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{(2k)^{2s+1}} \right\}. \end{aligned}$$

This is the required asymptotic expansion for $P_n^{(\alpha,\beta)}(z)$, $\alpha \geqq 0$.

Finally, we must compare the functions $\Pi_n^{(\alpha,\beta)}(z)$ and $W_2(\zeta)$. This is algebraically the most tedious of all the identifications so far considered. Suppose $\alpha > 0$ and not an integer; we note the following results. First, in the neighborhood of $z = 1$, we obtain from Eq. (1.2) that

$$(3.11) \quad (z - 1)^{-\alpha}(z + 1)^{-\beta}\Pi_n^{(\alpha,\beta)}(z) = -\frac{\pi\Gamma(n + \alpha + 1)}{\Gamma(n + 1)\Gamma(\alpha + 1)\sin(\pi\alpha)}$$

$$\times \left\{ 1 - \frac{2^\alpha\Gamma(\alpha)\Gamma(1 + \alpha)\Gamma(n + 1)\Gamma(n + \beta + 1)\sin(\pi\alpha)}{\pi\Gamma(n + \alpha + 1)\Gamma(n + \alpha + \beta + 1)(z - 1)^\alpha} \right\}\{1 + O(z - 1)\}.$$

Next, near $\zeta = 0$, since $A_s(\zeta) = \frac{1}{2}\zeta^2 A_s''(0) + O(\zeta^4)$ and $B_s(\zeta) = \zeta B_s'(0) + O(\zeta^3)$, we have

$$(3.12) \quad W_2(\zeta) \sim \frac{\pi}{2\sin(\pi\alpha)} \left\{ \frac{s(\alpha, K)}{k^\alpha\Gamma(1 - \alpha)}\zeta^{1-\alpha} - \frac{k^\alpha}{\Gamma(1 + \alpha)}\zeta^{1+\alpha} \right\}[1 + O(\zeta^2)],$$

where the function $s(\alpha, k)$ is defined by the asymptotic expansion

$$(3.13) \quad s(\alpha, k) \sim 1 - \frac{\alpha}{k}\sum_{s=0}^{\infty}\frac{B_s'(0)}{(2k)^{2s+1}}.$$

On comparing Eqs. (3.11) and (3.12), we find the required asymptotic expansion for $\Pi_n^{(\alpha,\beta)}(z)$ given by

$$(3.14) \quad \Pi_n^{(\alpha,\beta)}(z) \sim \frac{2^{(\alpha+\beta+3)/2}\Gamma(n + \alpha + 1)}{k^\alpha\Gamma(n + 1)}(z - 1)^{(2\alpha-1)/4}(z + 1)^{(2\beta-1)/4}$$

$$\times \left\{ \zeta^{1/2}K_\alpha(2k\zeta)\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{(2k)^{2s}} - \zeta^{1/2}K_{\alpha+1}(2k\zeta)\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{(2k)^{2s+1}} \right\}.$$

In addition, our comparison gives the curious result

$$(3.15) \quad \frac{k^{2\alpha}\Gamma(n + 1)\Gamma(n + \beta + 1)}{\Gamma(n + \alpha + 1)\Gamma(n + \alpha + \beta + 1)} \sim 1 - \frac{\alpha}{k}\sum_{s=0}^{\infty}\frac{B_s'(0)}{(2k)^{2s+1}}.$$

Equation (3.14) has been obtained under the assumption that $\alpha > 0$ and not an integer. The cases, when $\alpha = 0$ and α is a positive integer, must be treated separately since the functions then have different behavior near $z = 1$ or $\zeta = 0$ from that given in Eqs. (3.11) and (3.12). However, it can be shown that the asymptotic expansion given by Eq. (3.14) is in fact valid for all $\alpha \geqq 0$. The asymptotic expansion given by Eqs. (3.10) and (3.14) are believed to be new. They are uniformly valid for all z in the z-plane cut along the real axis from $-\infty$ to $+1$ with the neighborhood of the point $z = -1$ being deleted.

It may be noted that we can obtain similar expansions for $P_n^{(\alpha,\beta)}(z)$ and $\Pi_n^{(\alpha,\beta)}(z)$ which are valid in the neighborhood of the point $z = -1$ but not in the neighborhood of $z = 1$. These may be readily obtained from the above analysis by observing that

$$P_n^{(\alpha,\beta)}(-z) = (-1)^n P_n^{(\beta,\alpha)}(z),$$

and

$$\Pi_n^{(\alpha,\beta)}(-z) = (-1)^{n+1}\Pi_n^{(\beta,\alpha)}(z).$$

4. Relations Between the Coefficients a_s, A_s and B_s. In this section, we shall consider the purely algebraic relationships which exist between the coefficients $a_s(\zeta)$, as defined by Eqs. (2.12), and the coefficients $A_s(\zeta)$, $B_s(\zeta)$, as defined by Eqs. (3.6). The Eqs. (2.12) are of a rather simpler form than Eqs. (3.6), thus, we shall assume

that we know the coefficients $a_s(\zeta)$ and shall show how the coefficients $A_s(\zeta)$, $B_s(\zeta)$ may be obtained recursively from them. Our starting point is the observation that for ζ bounded away from $\zeta = 0$, the asymptotic expansions of $\Pi_n^{(\alpha,\beta)}(z)$, as given by Eqs. (2.14) and (3.14), must be the same. (We could compare the expansions for $P_n^{(\alpha,\beta)}(z)$, but the same results would be obtained.)

Now, for ζ bounded away from zero, we have that $|2k\zeta|$ is large when k is large. Since $|\arg \zeta| < \pi/2$, we have (see for example Erdélyi et al. [3, p. 86]) that

$$(4.1) \qquad \zeta^{1/2} K_\alpha(2k\zeta) \sim \frac{\pi^{1/2}}{2k^{1/2}}(z + (z^2 - 1)^{1/2})^{-k} \sum_{m=0}^{\infty} \frac{(\alpha, m)}{(4\zeta)^m k^m}.$$

Here, (α, m) denotes the "Hankel symbol" which is defined by

$$(4.2) \qquad (\alpha, m) = \frac{\Gamma(\tfrac{1}{2} + \alpha + m)}{m!\, \Gamma(\tfrac{1}{2} + \alpha - m)}, \quad \text{for } m = 0, 1, 2, \cdots.$$

On substituting this asymptotic expansion for $\zeta^{1/2}K_\alpha(2k\zeta)$ and $\zeta^{1/2}K_{\alpha+1}(2k\zeta)$ in Eq. (3.14) and comparing with Eq. (2.14), we find that

$$(4.3)$$
$$\frac{k^{\alpha+1/2}\Gamma[k + (\beta - \alpha + 1)/2]\Gamma[k - (\alpha + \beta - 1)/2]}{\Gamma(k + \tfrac{1}{2})\Gamma(k + 1)} \sum_{s=0}^{\infty} \frac{(-1)^s a_s(\zeta)}{2^s k^s}$$

$$= \left(\sum_{m=0}^{\infty} \frac{(\alpha, m)}{(4\zeta)^m k^m}\right)\left(\sum_{s=0}^{\infty} \frac{A_s(\zeta)}{2^{2s} k^{2s}}\right) - \left(\sum_{m=0}^{\infty} \frac{(\alpha + 1, m)}{(4\zeta)^m k^m}\right)\left(\sum_{s=0}^{\infty} \frac{B_s(\zeta)}{2^{2s+1} k^{2s+1}}\right).$$

We shall equate the coefficients of powers of $1/k$ on each side of this equation. First, we need a suitable expression for the ratio of the gamma functions appearing on the left-hand side of this equation. From a result given by Tricomi and Erdélyi [8] for the ratio of two gamma functions of large argument, we find

$$(4.4) \qquad \frac{k^{\alpha+1/2}\Gamma[k + (\beta - \alpha + 1)/2]\Gamma[k - (\alpha + \beta - 1)/2]}{\Gamma[k + \tfrac{1}{2}]\Gamma[k + 1]} \sim \sum_{l=0}^{\infty} \frac{c_l}{k^l}, \quad \text{for large } k$$

where the coefficients c_l are defined by

$$(4.5) \qquad c_l = \sum_{j=0}^{l} \frac{\Gamma[(\alpha + \beta + 1)/2 + j]\Gamma[(\alpha - \beta)/2 + l - j]}{j!\,(l - j)!\,\Gamma[(\alpha + \beta + 1)/2]\Gamma[(\alpha - \beta)/2]}$$
$$\cdot B_j^{[(1-\alpha-\beta)/2]}(0)B_{l-j}^{[1+(\beta-\alpha)/2]}(\tfrac{1}{2}),$$

for $l = 0, 1, 2, \cdots$. In this equation, $B_j^{[\sigma]}(x)$ denotes the generalised Bernoulli polynomial which is defined by

$$(4.6) \qquad \left(\frac{t}{e^t - 1}\right)^\sigma e^{xt} = \sum_{j=0}^{\infty} \frac{t^j}{j!}B_j^{[\sigma]}(x), \qquad |t| < 2\pi.$$

For properties of this polynomial, the reader is referred to Milne-Thomson [4]. On substituting the expansion (4.4) into Eq. (4.3), and equating powers of $1/k$ on each side of this equation, we find

$$(4.7)$$
$$\sum_{j=0}^{2l+1} \frac{(-1)^j a_j(\zeta)}{2^j} c_{2l+1-j}$$

$$= \sum_{j=0}^{l} \frac{[(\alpha, 2l - 2j + 1)A_j(\zeta) - 2\zeta B_j(\zeta)(\alpha + 1, 2l - 2j)]}{4^{2l-j+1}\zeta^{2l-2j+1}},$$

for $l = 0, 1, 2, \cdots$; and

$$(4.8) \quad \sum_{j=0}^{2l} \frac{(-1)^j a_j(\zeta)}{2^j} c_{2l-j}$$

$$= \sum_{j=0}^{l} \frac{[4\zeta A_j(\zeta)(\alpha, 2l - 2j) - 2B_{j-1}(\zeta)(\alpha + 1, 2l - 2j + 1)]}{4^{2l-j+1}\zeta^{2l-2j+1}} ,$$

for $l = 0, 1, 2, \cdots$, where $B_{-1}(\zeta) \equiv 0$. These two equations define recursively the coefficients $A_j(\zeta)$, $B_j(\zeta)$ in terms of $a_j(\zeta)$. Starting with $A_0(\zeta) = 1$, Eq. (4.7) with $l = 0$ gives $B_0(\zeta)$. With $l = 1$, Eq. (4.8) gives $A_1(\zeta)$ whence, from Eq. (4.7) with $l = 1$, we obtain $B_1(\zeta)$, etc.

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