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A New Method of Evaluation of Howland Integrals

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Abstract. In this paper, two Howland integrals are evaluated to 25D when the index is an odd integer. Those Howland integrals when the index is an even integer have been evaluated to 18D by Nelson. A new method of evaluation is used.

The four Howland integrals were first evaluated to 5D by Howland himself, partly with Stevenson, in the papers dealing with a perforated strip [1], [2]. Ling and Nelson in an earlier paper [3] evaluated these integrals to 6D by using a different method through some intermediate integrals. Later, Ling [4] reproduced the 6D values and also added values of a group of related integrals. Recently, Nelson [5], by using the same method, evaluated the integrals to 9D. In the process of computing some related integrals arising from axisymmetrical problems, Nelson, in the same paper, further evaluated the following two Howland integrals to 18D, when k is an even integer:

$$\begin{matrix} I_k \\ I_k^* \end{matrix} = \frac{1}{2(k!)} \int_0^\infty \frac{w^k\,dw}{\sinh w \pm w}, \qquad \begin{matrix} (k \geqq 1), \\ (k \geqq 3). \end{matrix} \tag{1}$$

The aim of the present paper is to evaluate these two integrals to 25D, when k is an odd integer, by using a direct method without recourse to the intermediate integrals. We begin by expanding the integrands into series as follows:

$$\begin{aligned} \frac{w^k}{\sinh w \pm w} &= \frac{2w^k e^{-w}}{1 \pm 2we^{-w} - e^{-2w}} \\ &= 2w^k e^{-w} \sum_{n=0}^{\infty} (\mp 1)^n p_n(w) e^{-nw}, \end{aligned} \tag{2}$$

where $p_n(w)$ is the Gegenbauer polynomial of degree n and order unity [6]. The expressions are found to be different depending on n being an even or an odd integer. They are, for $n \geqq 0$,

$$\begin{aligned} p_{2n}(w) &= \sum_{m=0}^{n} (-1)^{n+m} \binom{n+m}{2m} (2w)^{2m}, \\ p_{2n+1}(w) &= \sum_{m=0}^{n} (-1)^{n+m} \binom{n+m+1}{2m+1} (2w)^{2m+1}. \end{aligned} \tag{3}$$

With the aid of the integral

$$\int_0^\infty w^m e^{-aw}\,dw = \frac{m!}{a^{m+1}}, \qquad (a > 0) \tag{4}$$

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the following series are obtained for any integral value of k under the restrictions as indicated,

$$\begin{matrix} I_k \\ I_k^* \end{matrix} = 1 \mp \frac{q_2(k)}{2^{k+2}} + \frac{q_3(k)}{3^{k+3}} \mp \frac{q_4(k)}{4^{k+4}} + \frac{q_5(k)}{5^{k+5}} \mp \cdots \qquad \begin{matrix} (k \geqq 1) \\ (k \geqq 3) \end{matrix}, \tag{5}$$

where

$$\begin{aligned} q_2(k) &= 2(k+1), \\ q_3(k) &= 4k^2 + 12k - 1, \\ q_4(k) &= 8(k+1)(k^2 + 5k - 2), \\ q_5(k) &= 16k^4 + 160k^3 + 260k^2 - 100k + 409. \end{aligned} \tag{6}$$

In general, $q_n(k)$ is a polynomial of k of degree $n - 1$ with integral coefficients. The general expressions are

$$\begin{aligned} q_{2n+1}(k) &= \sum_{m=0}^{n} (-1)^{n+m} \binom{2m+k}{k} \frac{(n+m)!}{(n-m)!} 2^{2m}(2n+1)^{2n-2m}, \\ q_{2n+2}(k) &= \sum_{m=0}^{n} (-1)^{n+m} \binom{2m+k+1}{k} \frac{(n+m+1)!}{(n-m)!} 2^{2m+1}(2n+2)^{2n-2m}. \end{aligned} \tag{7}$$

The preceding series for the integrals are rapidly convergent when k is a large integer but slowly convergent when k is a small integer. In particular, the series for I_1 and I_2 are believed to be only conditionally convergent. For instance, an accuracy of 25D can be attained with only the first five terms for $k \geqq 44$, ten terms for $k \geqq 33$ and as many as fifty terms for $k \geqq 20$. Hence, it is necessary to use some other method to evaluate the early integrals. By a combined use of Cauchy's integral theorem and Cauchy's theorem of residues, the integrals I_{2k-1} and I_{2k-1}^* are developed into series as follows: For $k \geqq 1$,

$$\begin{aligned} I_{2k-1} &= \frac{1}{2(2k-1)!} \left\{ \frac{\alpha}{4} \delta_{k,1} + \alpha \sum_{n=1}^{\infty} \frac{(n\alpha)^{2k-1}}{\sinh n\alpha + n\alpha} \right. \\ &\qquad \left. + \operatorname{Re} \sum_{m=1}^{\infty} \frac{\pi z_m^{2k-1} \exp(\pi i z_m/\alpha)}{\cosh^2(z_m/2) \sin(\pi z_m/\alpha)} \right\} \\ I_{2k-1} &= \frac{1}{2(2k-1)!} \left\{ \alpha \sum_{n=0}^{\infty} \frac{(n\alpha + \frac{1}{2}\alpha)^{2k-1}}{\sinh(n+\frac{1}{2})\alpha + (n+\frac{1}{2})\alpha} \right. \\ &\qquad \left. + \operatorname{Re} \sum_{m=1}^{\infty} \frac{\pi i z_m^{2k-1} \exp(\pi i z_m/\alpha)}{\cosh^2(z_m/2) \cos(\pi z_m/\alpha)} \right\} \end{aligned} \tag{8}$$

and for $k \geqq 2$,

$$\begin{aligned} I_{2k-1}^* &= \frac{1}{2(2k-1)!} \left\{ 3\alpha\, \delta_{k,2} + \alpha \sum_{n=1}^{\infty} \frac{(n\alpha)^{2k-1}}{\sinh n\alpha - n\alpha} \right. \\ &\qquad \left. + \operatorname{Re} \sum_{m=1}^{\infty} \frac{\pi (z_m^*)^{2k-1} \exp(\pi i z_m^*/\alpha)}{\sinh^2(z_m^*/2) \sin(\pi z_m^*/\alpha)} \right\} \\ I_{2k-1}^* &= \frac{1}{2(2k-1)!} \left\{ \alpha \sum_{n=0}^{\infty} \frac{(n\alpha + \frac{1}{2}\alpha)^{2k-1}}{\sinh(n+\frac{1}{2})\alpha - (n+\frac{1}{2})\alpha} \right. \\ &\qquad \left. + \operatorname{Re} \sum_{m=1}^{\infty} \frac{\pi i (z_m^*)^{2k-1} \exp(\pi i z_m^*/\alpha)}{\sinh^2(z_m^*/2) \cos(\pi z_m^*/\alpha)} \right\}, \end{aligned} \tag{9}$$

where α is a positive constant, $\delta_{k,n}$ is the Kronecker delta, and z_m and z_m^* are the mth complex zeros of $\sinh z \pm z$, respectively, in the first quadrant of the z-plane. The results give two different expressions for each integral. The derivation will be described later. Each integral can then be computed from one expression and checked by the other.

It is seen that each expression consists of two series. The first one is a real series and the second the real part of a complex series. It is also seen that the constant α occurs only on the right-hand side of each expression. This constant can be fixed to suit our convenience. The first series converges rapidly when α is large and the second when α is small. In fact, the first series of the first expression of each integral represents the value given by the trapezoidal rule, and the first series of the second expression represents the value given by the rectangular rule. In both cases, α is the width of the strip. Therefore, the second series of each expression may be regarded merely as a correction, analogous to the second series in Gregory's formula [7]. By a proper choice of α, the value of the second series can generally be made small in comparison with that of the first series.

In the computation, the value of α is taken as unity. Unlike the series in (5), the convergence of the first series becomes slower as k increases. To attain an accuracy of 25D with this value of α, 65 terms of the first series are needed for $k = 1$, 130 terms for $k = 15$ and 200 terms for $k = 35$. The corresponding numbers of terms needed to attain an accuracy of 18D are 47, 110 and 175, respectively. The number of terms needed in each instance decreases to one half if the value of α is doubled. In computing the second series, the 11D values of z_m and z_m^* computed before by Ling [8] are available. Their accuracy can be improved readily, whenever needed, by using the Newton-Raphson method. The convergence of the second series is so rapid that when $\alpha = 1$, at most two terms are needed for the present computation.

An alternative method for computing these values is to attempt to evaluate the remainder term in (5). This remainder term has an integral representation of a similar nature as I_k or I_k^* itself, but the integrand is more complicated. On the other hand, when the Gauss-Laguerre quadrature rule was used in the evaluation, it was found that, for small values of k, adequate precision could not be obtained without an effort far exceeding that required when using (8) and (9).

The computation was carried out on an IBM 1620 computer. The following relations were used as a further check:

$$\sum_{k=1}^{\infty} (1 - I_{2k-1}) = I_1 - \tfrac{1}{4}, \qquad (10)$$

$$\sum_{k=2}^{\infty} (I_{2k-1}^* - 1) = \tfrac{5}{4}.$$

Some typical values of I_k and I_k^* rounded to 25D are shown in the accompanying table. The complete results for odd integral values of k up to 91 appear in the tables on pp. 334 and 335 of this paper.

Derivation of Expressions (8) *and* (9). Consider the contour integral

$$\frac{1}{2\pi i} \oint \frac{z^{2k-1}\, dz}{(z - t)(\sinh z + z) \cos (\pi z/\alpha)}, \qquad (k \geqq 1), \qquad (11)$$

Table. Howland integrals I_k and I_k^* when k is an odd integer

k	I_k	I_k^*
1	0.76857 45381 11553 68096 76880	∞
3	0.82771 02958 85164 81343 56543	2.03871 06665 65932 70071 50016
5	0.92547 59977 84897 73778 40994	1.15686 43660 75341 56854 09629
7	0.97269 89930 38011 36576 56831	1.03925 13121 81494 86407 63193
9	0.99094 91791 22157 96577 36401	1.01087 01465 33325 36251 87849
11	0.99718 85753 87716 81086 85869	1.00308 47737 70445 49356 36804
13	0.99916 38232 58856 72120 94672	1.00087 61801 93056 13024 90264
15	0.99975 85587 66695 28941 74027	1.00024 71389 53388 81695 24967
17	0.99993 17185 56121 05194 10886	1.00006 90733 57675 51266 51905
19	0.99998 09793 21140 41769 14442	1.00001 91284 43795 33176 43980
21	0.99999 47619 05105 76299 10821	1.00000 52524 98197 16132 17372
23	0.99999 85704 29417 28762 07253	1.00000 14314 66414 40027 00505
25	0.99999 96126 92405 13473 89937	1.00000 03875 53890 75565 46225
27	0.99999 98957 07648 54992 23669	1.00000 01043 23990 59043 58713
29	0.99999 99720 62331 92616 96717	1.00000 00279 41692 90959 92369
31	0.99999 99925 49672 97929 12797	1.00000 00074 50834 66324 65526
33	0.99999 99980 20971 28404 08009	1.00000 00019 79092 26267 56463
35	0.99999 99994 76135 00127 56009	1.00000 00005 23872 90041 29141
37	0.99999 99998 61757 29356 59978	1.00000 00001 38243 68310 21172
39	0.99999 99999 63620 27195 72193	1.00000 00000 36379 84811 33639
41	0.99999 99999 90450 31297 57245	1.00000 00000 09549 70171 39873
43	0.99999 99999 97498 89046 47118	1.00000 00000 02501 11132 45368
45	0.99999 99999 99346 30079 16274	1.00000 00000 00653 69942 54313
47	0.99999 99999 99829 46975 65349	1.00000 00000 00170 53026 97001
49	0.99999 99999 99955 59108 05951	1.00000 00000 00044 40892 25651

Table. Howland integrals I_k and I_k^* when k is an odd integer (cont'd)

k	I_k	I_k^*
51	0.99999 99999 99988 45368 07336	1.00000 00000 00011 54631 96458
53	0.99999 99999 99997 00239 78562	1.00000 00000 00002 99760 21892
55	0.99999 99999 99999 22284 38855	1.00000 00000 00000 77715 61199
57	0.99999 99999 99999 79877 20771	1.00000 00000 00000 20122 79235
59	0.99999 99999 99999 94795 82958	1.00000 00000 00000 05204 17043
61	0.99999 99999 99999 98655 58931	1.00000 00000 00000 01344 41069
63	0.99999 99999 99999 99653 05530	1.00000 00000 00000 00346 94470
65	0.99999 99999 99999 99910 55332	1.00000 00000 00000 00089 44668
67	0.99999 99999 99999 99976 96070	1.00000 00000 00000 00023 03930
69	0.99999 99999 99999 99994 07077	1.00000 00000 00000 00005 92923
71	0.99999 99999 99999 99998 47534	1.00000 00000 00000 00001 52466
73	0.99999 99999 99999 99999 60825	1.00000 00000 00000 00000 39175
75	0.99999 99999 99999 99999 89941	1.00000 00000 00000 00000 10059
77	0.99999 99999 99999 99999 97419	1.00000 00000 00000 00000 02581
79	0.99999 99999 99999 99999 99338	1.00000 00000 00000 00000 00662
81	0.99999 99999 99999 99999 99830	1.00000 00000 00000 00000 00170
83	0.99999 99999 99999 99999 99957	1.00000 00000 00000 00000 00043
85	0.99999 99999 99999 99999 99989	1.00000 00000 00000 00000 00011
87	0.99999 99999 99999 99999 99997	1.00000 00000 00000 00000 00003
89	0.99999 99999 99999 99999 99999	1.00000 00000 00000 00000 00001
91	1.00000 00000 00000 00000 00000	1.00000 00000 00000 00000 00000

TABLE

k	I_k	I_k^*
1	0.76857 45381 11553 68096 76880	∞
3	0.82771 02958 85164 81343 56543	2.03871 06665 65932 70071 50016
5	0.92547 59977 84897 73778 40994	1.15686 43660 75341 56854 09629
15	0.99975 85587 66695 28941 74027	1.00024 71389 53388 81695 24967
25	0.99999 96126 92405 13473 89937	1.00000 03875 53890 75565 46225
35	0.99999 99994 76135 00127 56009	1.00000 00005 23872 90041 29141
55	0.99999 99999 99999 22284 38855	1.00000 00000 00000 77715 61199
85	0.99999 99999 99999 99999 99989	1.00000 00000 00000 00000 00011

where the contour is taken round the circle $|z| = R$ through a sequence of values such that the circle never passes through any pole of the integrand, t being any point on the x axis inside the circle and α a positive constant. The integral tends to zero as R tends to infinity. The poles of the integrand are

$$z = t, \quad z = \pm z_m, \quad z = \pm \bar{z}_m, \quad z = \pm(n + \tfrac{1}{2})\alpha$$

where $m = 1, 2, 3, \cdots, n = 0, 1, 2, \cdots$, and a bar denotes the complex conjugate, z_m being defined before. Note that the complex zeros of $\sinh z + z$ in the entire z-plane are symmetrically located in each quadrant with respect to both the x and y axes. Furthermore, the origin $z = 0$ is also a zero. Both poles are of order unity.

It follows from Cauchy's theorems that the sum of residues at all the poles is zero. Consequently, we find

$$\begin{aligned}\frac{t^{2k-1}}{(\sinh t + t)\cos(\pi t/\alpha)} &= \frac{\alpha}{\pi}\sum_{n=0}^{\infty}\frac{2(-1)^n(n\alpha + \frac{1}{2}\alpha)^{2k}}{\sinh(n+\frac{1}{2})\alpha + (n+\frac{1}{2})\alpha}\cdot\frac{1}{(n\alpha+\frac{1}{2}\alpha)^2 - t^2} \\ &\quad - \operatorname{Re}\sum_{m=1}^{\infty}\frac{2z_m^{2k}}{\cosh^2(z_m/2)\cos(\pi z_m/\alpha)}\cdot\frac{1}{z_m^2 - t^2}.\end{aligned} \tag{12}$$

Multiplying by $\cos(\pi t/\alpha)$, integrating with respect to t from zero to infinity, and making use of the following integrals

$$\begin{aligned}\int_0^\infty \frac{\cos(\pi t/\alpha)\,dt}{(n\alpha + \frac{1}{2}\alpha)^2 - t^2} &= \frac{(-1)^n\pi}{(2n+1)\alpha}, \\ \int_0^\infty \frac{\cos(\pi t/\alpha)\,dt}{z_m^2 - t^2} &= -\frac{\pi i}{2z_m}\exp(\pi i z_m/\alpha),\end{aligned} \tag{13}$$

we find the second expression in (8).

Again, consider the contour integral

$$\frac{1}{2\pi i}\oint \frac{z^{2k-1}\,dz}{(z-t)(\sinh z + z)\sin(\pi z/\alpha)}, \qquad (k \geqq 1). \tag{14}$$

The poles of the integrand are

$$z = t, \quad z = \pm z_m, \quad z = \pm \bar{z}_m, \quad z = \pm n\alpha,$$

where $m = 1, 2, 3, \cdots$ and $n = 1, 2, 3, \cdots$. In particular, when $k = 1$, an additional pole is at the origin $z = 0$. By making use of the following integrals

$$\int_0^\infty \frac{t \sin(\pi t/\alpha)\, dt}{(n\alpha)^2 - t^2} = -\frac{(-1)^n \pi}{2}, \tag{15}$$
$$\int_0^\infty \frac{t \sin(\pi t/\alpha)\, dt}{z_m^2 - t^2} = -\frac{\pi}{2} \exp(\pi i z_m/\alpha),$$

we find similarly the first expression in (8).

By replacing $\sinh z + z$ with $\sinh z - z$ in the foregoing two contour integrals, we likewise find the two expressions in (9) for $k \geqq 2$.

It should be mentioned that two expressions for each I_{2k} and I_{2k}^* can be derived in a similar manner. However, the resulting expressions appear to be less simple since they also involve sine and cosine integrals. It should also be mentioned that the foregoing method of evaluation can be generalized.

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