MATHEMATICS OF COMPUTATION, VOLUME 25, NUMBER 114, APRIL, 1971

Calculation of the Gamma Function by Stirling's Formula

By Robert Spira

Abstract. In this paper, we derive a simple error estimate for the Stirling formula and also give numerical coefficients.

Stirling's formula is:

$$\log \Gamma(s) = (s - \tfrac{1}{2}) \log s - s + \tfrac{1}{2} \log 2\pi + \sum_{k=1}^{m} s^{1-2k}(2k)^{-1}(2k-1)^{-1}B_{2k} + R_m \tag{1}$$

where

$$R_m = -\int_0^\infty (s+x)^{-2m}(2m)^{-1}B_{2m}(x - [x])\, dx. \tag{2}$$

Formulas (1) and (2) and a simple estimate for $|R_m|$ are derived in de Bruijn [1, pp. 46–48].

Another form of R_m, developed on the assumption Re $s > 0$, is

$$R_m = \frac{2(-1)^m}{s^{2m-1}} \int_0^\infty \left\{ \int_0^t \frac{u^{2m}\, du}{u^2 + s^2} \right\} \frac{dt}{e^{2\pi t} - 1},$$

(Whittaker and Watson [5, p. 252]), and Whittaker and Watson also estimate this expression, finding

$$|R_m| \leqq \frac{|B_{2m+2}|\, K(s)}{(2m+1)(2m+2)\, |s|^{2m+1}}$$

where

$$K(s) = \text{upper bound } |s^2/(u^2+s^2)|, \qquad u \geqq 0.$$

This is the form given in the NBS *Handbook*, and is clearly poor near the imaginary axis. It follows, however, from this form, that if $|\arg s| \leqq \pi/4$, then the error in taking the first m terms of the asymptotic series is less in absolute value than the absolute value of the $(m+1)$st term. Another form of the remainder, valid for $|\arg s| \leqq \pi - \delta$, is derived in Whittaker and Watson [5, §13.6], but this remainder involves the Hurwitz zeta function, and has never been used for numerical estimates. An estimate for R_m, as given by (2), may be found in Nielsen [6, p. 208], and, expressed in current notation, is

$$|R_m(s)| < \frac{|B_{2m+2}|}{(2m+1)(2m+2)\, |s|^{2m+1} (\cos(\tfrac{1}{2} \arg s))^{2m+2}}.$$

Received May 15, 1970, revised November 19, 1970.
AMS 1969 *subject classifications.* Primary 6525.
Key words and phrases. Asymptotic series, gamma function.

This gives a uniform estimate in the angle $|\arg s| \leqq \pi - \delta$. We now develop an estimate for R_m which has the advantages of simplicity in application, and uniformity for a set of points whose distance from the negative real axis is $\geqq$ some fixed amount.

THEOREM.

$$(3)\qquad |R_m| \leqq 2\,|B_{2m}/(2m-1)|\cdot|\operatorname{Im} s|^{1-2m} \qquad \text{for } \operatorname{Re} s < 0,\ \operatorname{Im} s \neq 0,$$

$$(4)\qquad |R_m| \leqq |B_{2m}/(2m-1)|\cdot|s|^{1-2m} \qquad \text{for } \operatorname{Re} s \geqq 0.$$

Proof. Since $B_{2m}(x-[x])$ varies only slightly over the range of x, and $|B_{2m}(x-[x])| \leqq |B_{2m}|$, the problem of estimating $|R_m|$ reduces to the problem of estimating $\int_0^\infty |s+x|^{-2m}\,dx$. Note that the integrand will be large only when s is near $-x$. By symmetry, we need only consider the case when $\operatorname{Im} s \geqq 0$. First, let $\operatorname{Re} s < 0$ and $\operatorname{Im} s \neq 0$. Then, taking $k = \operatorname{Im} s$,

$$\int_0^\infty |s+x|^{-2m}\,dx = \int_0^{-\operatorname{Re} s} + \int_{-\operatorname{Re} s}^{-\operatorname{Re} s+k} + \int_{-\operatorname{Re} s+k}^{\infty}.$$

Estimating the integrands of the second integral by $|s+x|^{-2m} \leqq k^{-2m}$, and of the third by $|s+x|^{-2m} \leqq (x+\operatorname{Re} s)^{-2m}$, we obtain

$$\int_0^\infty |s+x|^{-2m}\,dx \leqq \int_0^{-\operatorname{Re} s} |s+x|^{-2m}\,dx + k^{1-2m} + (2m-1)^{-1}(k)^{1-2m}.$$

It remains to estimate $\int_0^{-\operatorname{Re} s}$. If $-\operatorname{Re} s \leqq k$, we approximate the integrand again by k^{-2m}, giving

$$\int_0^{-\operatorname{Re} s} |s+x|^{-2m}\,dx \leqq (-\operatorname{Re} s)\cdot k^{-2m} \leqq k^{1-2m}.$$

If, however, $-\operatorname{Re} s > k$, we break up the range of integration, giving

$$\begin{aligned}\int_0^{-\operatorname{Re} s} |s+x|^{-2m}\,dx &\leqq \int_0^{-\operatorname{Re} s-k} |s+x|^{-2m}\,dx + \int_{-\operatorname{Re} s-k}^{-\operatorname{Re} s} |s+x|^{-2m}\,dx \\ &\leqq \int_0^{-\operatorname{Re} s-k} (-x-\operatorname{Re} s)^{-2m}\,dx + k^{1-2m} \\ &= \frac{1}{2m-1}\,[k^{1-2m} - (-\operatorname{Re} s)^{1-2m}] + k^{1-2m} \\ &\leqq (1+1/(2m-1))k^{1-2m}.\end{aligned}$$

So that in all cases, if $\operatorname{Re} s < 0$

$$\int_0^\infty |x+s|^{-2m}\,dx \leqq (4m/(2m-1))k^{1-2m},$$

so we have derived (3).

If $\operatorname{Re} s \geqq 0$, then $|s+x|^{-2m} \leqq |ki+x|^{-2m}$ since

$$|s+x|^2 = (\operatorname{Re} s + x)^2 + (\operatorname{Im} s)^2 = 2x\operatorname{Re} s + x^2 + k^2 \geqq |ki+x|^2.$$

Next, estimating as before,

$$\int_0^\infty |ki + x|^{-2m}\,dx \leqq \int_0^k k^{-2m}\,dx + \int_k^\infty x^{-2m}\,dx \leqq k^{1-2m}(1 + 1/(2m - 1)),$$

thus giving (4), and completing the proof.

On taking the exponential, we find

$$\Gamma(s) \sim (2\pi)^{1/2} e^{-s} s^{s-1/2} \exp\left[\sum_{k=1}^{N_1} \frac{A_{2k-1}}{s^{2k-1}}\right] \tag{5}$$

where

$$A_{2k-1} = B_{2k}/2k(2k - 1). \tag{6}$$

A short calculation gives (formally)

$$\begin{aligned}\exp\left[\sum_{k=1}^{\infty} \frac{A_{2k-1}}{s^{2k-1}}\right] &= 1 + \sum_{k=1}^{\infty} s^{-k}\left[\sum_{(\alpha_1 i_1, \cdots, \alpha_n i_n) \in Q(k)} \frac{A_{\alpha_1}^{i_1} A_{\alpha_2}^{i_2}, \cdots, A_{\alpha_n}^{i_n}}{j_1!\ j_2!\ , \cdots, j_n!}\right] \\ &= 1 + \sum_{k=1}^{\infty} c_k s^{-k}\end{aligned} \tag{7}$$

where the α_i's are distinct and $Q(k)$ is the set of partitions of k into odd parts ($\alpha_i^{j_i}$ means α_i repeated j_i times in the partition).

Wrench [2] found the recurrences

$$(2k - 1)c_{2k-1} = \frac{B_2}{2} c_{2k-2} + \frac{B_4}{4} c_{2k-4} + \cdots + \frac{B_{2k}}{2k}, \tag{8}$$

$$2kc_{2k} = \frac{B_2}{2} c_{2k-1} + \frac{B_4}{4} c_{2k-3} + \cdots + \frac{B_{2k}}{2k} c_1, \tag{9}$$

where $k = 1, 2, 3, \cdots$ and $c_0 = 1$, and these formulas are more suitable for calculation than (7).

Wrench [2] also gave the c_j's for $j = 0(1)20$, in exact form and to 50D, and also found approximations to about 6S for $j = 21(1)30$. We give in Table 1 the exact rational values for $j = 21(1)30$, and in Table 2 their 45D equivalents. The following corrections are necessary in Wrench's tables. In his Table 2, the last ten digits of c_{13} read 01893 93280, and should read 01894 09396. In his Table 3, entries 22, 23, 24, 26, 28, 30 can be corrected from Table 2 of this paper. Dr. Wrench confirmed the correctness of the author's value for c_{13}, and that it is likely that the author's corrections to his Table 3 are also valid. It is of interest to note that while Dr. Wrench's calculations were carried out on a desk calculator, the author's were performed on a Fortran simulator of a large decimal machine (Spira [7]).

A further calculation revealed that entries 3, 4, 7, 8, 11, 12, 15, 16, 17 for c_{n+1}/c_n in Table XII of Spira [3] have errors beyond 16S. These errors did not affect the remaining tables.

Finally, we remark that estimates for the error in using

$$\Gamma(s) \sim (2\pi)^{1/2} e^{-s} s^{s-1/2}\left\{1 + \frac{c_1}{s} + \frac{c_2}{s^2} + \cdots + \frac{c_k}{s^k}\right\} \tag{10}$$

can be obtained from estimating

$$\exp\left\{\sum_{j=1}^{m} A_{2j-1} s^{1-2j} + R_m\right\} - \sum_{j=1}^{k} c_j s^{-j} \tag{11}$$

TABLE 1

Coefficients 21 through 30 in the Stirling asymptotic series for $\Gamma(z)$

c_{21}		34856 85173 42344 01648 33562 31076 88675 64083 96794 47003 / 2601 64872 18125 16297 62664 73959 14866 28167 68000 00000
c_{22}		9 09773 12459 95425 06852 27522 94225 93983 24288 04521 45053 / 8 11714 40120 55050 84859 51398 75254 38279 88316 16000 00000
c_{23}	−	15273 35577 85467 70230 23224 27280 09471 25313 62926 72693 90501 / 97 40572 81446 60610 18314 16785 03052 59358 59793 92000 00000
c_{24}	−	18 38564 55668 17780 20033 16143 79951 80647 19008 29995 86348 26921 / 1 40264 24852 83112 78663 72401 70443 95734 76381 03244 80000 00000
c_{25}		2 58331 20988 61137 96374 59020 36370 49694 38721 38148 65171 20938 16393 / 117 82196 87637 81474 07752 81743 17292 41720 16006 72563 20000 00000
c_{26}		5 18013 42908 22682 44375 77104 27952 46758 19182 33549 14089 67023 64013 / 2827 72725 03307 55377 86067 61836 15018 01283 84161 41516 80000 00000
c_{27}	−	5275 50309 09787 33965 92733 54057 99289 93424 14298 36915 19876 84094 84184 33873 / 14613 12888 42592 77641 70840 23816 85690 08674 63669 36130 51904 00000 00000
c_{28}	−	21148 66241 53708 11646 13223 32421 55728 12504 64870 36484 82437 46060 29560 15127 / 7 01430 18644 44453 26802 00331 43209 13124 16382 56129 34264 91392 00000 00000
c_{29}		18039 44129 15538 78214 00157 77241 22802 51037 85450 23572 62351 75126 98174 30990 27459 / 2609 32029 35733 36615 70345 23292 73796 82188 94312 80115 46547 97824 00000 00000
c_{30}		3226 14019 20539 36286 91281 19490 56082 64758 66044 17173 68772 94520 86326 36420 80203 03641 / 5589 16406 88340 87030 83679 48893 04472 79248 71618 02007 32705 76939 00800 00000 00000

TABLE 2

45D Values of coefficients 21 through 30 in the asymptotic series for $\Gamma(z)$

c_{21}	13.39798	54551	42589	21762	69304	32019	67195	04205	85565
c_{22}	1.12080	44642	89911	60686	26394	00139	92394	10087	44581
c_{23}	−156.80141	27040	22726	37282	36984	46041	18986	42959	25353
c_{24}	−13.10786	30226	33865	65902	75053	22267	17265	62139	54267
c_{25}	2192.55553	60905	23432	96901	29668	35404	98912	17444	39338
c_{26}	183.19073	34845	24338	08866	21120	60475	26830	49008	10167
c_{27}	−36101.11929	32220	75951	91379	10143	10212	31172	74408	12019
c_{28}	−3015.07731	26223	05854	21582	73842	95134	58512	61670	77656
c_{29}	691346.37614	18781	21600	20149	42362	07859	56471	17679	20033
c_{30}	57721.33636	30407	22716	58721	99716	32365	57540	83996	54732

and using (3) and (4), where $m = [(k+2)/2]$. For example, for $\operatorname{Re} s \geqq 0$ and $|s| \geqq 1$, and $k = m = 2$, we have

$$\Gamma(s) = (2\pi)^{1/2} e^{-s} s^{s-1/2} \exp\left\{\frac{1}{12s} + \frac{1}{360s^3} + R_2\right\},$$

where

$$|R_2| \leqq \frac{1}{90|s|^3},$$

so

$$|\exp R_2 - 1| \leqq |R_2| \, \{1 + |R_2| + |R_2|^2 + \cdots\} \leqq \frac{1}{89|s|^3}.$$

Next,

$$\left|\exp\left(\frac{1}{12s} + \frac{1}{360s^3}\right) - \left(1 + \frac{1}{12s} + \frac{1}{288s^2}\right)\right|$$

$$\leqq \frac{1}{360|s|^3} + \frac{1}{12\cdot 360|s|^4} + \frac{1}{2\cdot 360^2|s|^6} + \frac{1}{3!}\left|\frac{1}{12s} + \frac{1}{360s^3}\right|^3 + \cdots$$

which estimates as before. Such estimates show the series for $\Gamma(s)$ is an asymptotic series (de Bruijn [1]).

For calculations near the origin, it is best to use the functional equation $\Gamma(s+1) = s\Gamma(s)$ and calculate $\Gamma(s) = \Gamma(s+j)/P(s)$, where $P(s)$ is a polynomial. This formula could also be used for larger $|s|$ for ultraprecise calculations where precisions are needed which are greater than the maximum precision obtainable from the asymptotic formula. For calculations in the left half-plane with small imaginary part, one can use the equation $\Gamma(s)\Gamma(1 - s) = \pi/\sin \pi s$.

The preparation of this paper was with the aid of NSF Grant GP-8957. I wish to thank the referee for several suggestions.

Mathematics Department
Michigan State University
East Lansing, Michigan 48823

1. N. G. de Bruijn, *Asymptotic Methods in Analysis*, Bibliotheca Math., vol. 4, North-Holland, Amsterdam; Noordhoff, Groningen; Interscience, New York, 1958. MR **20** #6003.
2. J. W. Wrench, Jr., "Concerning two series for the gamma function," *Math. Comp.*, v. 22, 1968, pp. 617–626. MR **38** #5371.
3. R. Spira, *Table of the Riemann Zeta Function*, UMT files, reviewed in *Math. Comp.*, v. 18, 1964, pp. 519–521.
4. *Table of the Gamma Function for Complex Arguments*, Nat. Bur. Standards, Appl. Math. Series, vol. 34, 1954.
5. E. T. Whittaker & G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge Univ. Press, New York, 1962. MR **31** #2375.
6. N. Nielsen, *Die Gammafunction*. Band I. *Handbuch der Theorie der Gammafunktion*. Band II. *Theorie des Integrallogarithmus und verwandter Transzendenten*, Chelsea, New York, 1965. MR **32** #2622.
7. R. Spira, *Fortran Multiple Precision*. Parts I, II, Mathematics Department, Michigan State University, East Lansing, Michigan, 1970.