Calculation of the Gamma Function by Stirling’s Formula

By Robert Spira

Abstract. In this paper, we derive a simple error estimate for the Stirling formula and also give numerical coefficients.

Stirling’s formula is:

$$\log \Gamma(s) = (s - \frac{1}{2}) \log s - s + \frac{1}{2} \log 2\pi + \sum_{k=1}^{m} \frac{s^{-2k}(2k)^{-1}(2k - 1)^{-1}B_{2k}}{s^{2k}} + R_m$$

where

$$R_m = -\int_{0}^{m} (s + x)^{-2m}(2m)^{-1}B_{2m}(x - [x]) \, dx.$$ 

Formulas (1) and (2) and a simple estimate for $|R_m|$ are derived in de Bruijn [1, pp. 46-48].

Another form of $R_m$, developed on the assumption $\text{Re } s > 0$, is

$$R_m = \frac{2(-1)^m}{s^{2m-1}} \int_{0}^{s} \left( \int_{0}^{t} \frac{u^{2m} \, du}{u^2 + s^2} \right) \frac{dt}{e^{2\pi t} - 1},$$

(Whittaker and Watson [5, p. 252]), and Whittaker and Watson also estimate this expression, finding

$$|R_m| \leq \frac{|B_{2m+2}| K(s)}{(2m + 1)(2m + 2) |s|^{2m+1}}$$

where

$$K(s) = \text{upper bound } |s^2/(u^2 + s^2)|, \quad u \geq 0.$$ 

This is the form given in the NBS Handbook, and is clearly poor near the imaginary axis. It follows, however, from this form, that if $|\arg s| \leq \pi/4$, then the error in taking the first $m$ terms of the asymptotic series is less in absolute value than the absolute value of the $(m + 1)$st term. Another form of the remainder, valid for $|\arg s| \leq \pi - \delta$, is derived in Whittaker and Watson [5, §13.6], but this remainder involves the Hurwitz zeta function, and has never been used for numerical estimates. An estimate for $R_m$, as given by (2), may be found in Nielsen [6, p. 208], and, expressed in current notation, is

$$|R_m(s)| < \frac{|B_{2m+2}|}{(2m + 1)(2m + 2) |s|^{2m+1}} (\cos \left( \frac{1}{2} \arg s \right))^{2m+2}.$$ 

Received May 15, 1970, revised November 19, 1970.

AMS 1969 subject classifications. Primary 6525.

Key words and phrases. Asymptotic series, gamma function.

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This gives a uniform estimate in the angle $|\arg s| \leq \pi - \delta$. We now develop an estimate for $R_m$ which has the advantages of simplicity in application, and uniformity for a set of points whose distance from the negative real axis is $\geq$ some fixed amount.

**Theorem.**

(3) $|R_m| \leq 2 |B_{2m}/(2m - 1)| \cdot |\text{Im } s|^{1-2m}$ for $\text{Re } s < 0$, $\text{Im } s \neq 0$,

(4) $|R_m| \leq |B_{2m}/(2m - 1)| \cdot |s|^{1-2m}$ for $\text{Re } s \geq 0$.

**Proof.** Since $B_{2m}(x - [x])$ varies only slightly over the range of $x$, and $|B_{2m}(x - [x])| \leq |B_{2m}|$, the problem of estimating $|R_m|$ reduces to the problem of estimating $\int_0^\infty |s + x|^{-2m} \, dx$. Note that the integrand will be large only when $s$ is near $-x$. By symmetry, we need only consider the case when $\text{Im } s \geq 0$. First, let $\text{Re } s < 0$ and $\text{Im } s \neq 0$. Then, taking $k = \text{Im } s$,

$$\int_0^\infty |s + x|^{-2m} \, dx = \int_0^{-\text{Re } s} + \int_{-\text{Re } s}^{-\text{Re } s + k} + \int_{s + x|^{-2m}}.$$ 

Estimating the integrands of the second integral by $|s + x|^{-2m} \leq k^{-2m}$, and of the third by $|s + x|^{-2m} \leq (x + \text{Re } s)^{-2m}$, we obtain

$$\int_0^\infty |s + x|^{-2m} \, dx \leq \int_0^{-\text{Re } s} |s + x|^{-2m} \, dx + k^{-2m} + (2m - 1)^{1-2m}.$$ 

It remains to estimate $\int_{-\text{Re } s}^{-\text{Re } s}$. If $-\text{Re } s \leq k$, we approximate the integrand again by $k^{-2m}$, giving

$$\int_0^{-\text{Re } s} |s + x|^{-2m} \, dx \leq (-\text{Re } s) \cdot k^{-2m} \leq k^{1-2m}.$$ 

If, however, $-\text{Re } s > k$, we break up the range of integration, giving

$$\int_0^{-\text{Re } s} |s + x|^{-2m} \, dx \leq \int_0^{-\text{Re } s - k} |s + x|^{-2m} \, dx + \int_{-\text{Re } s - k}^{-\text{Re } s} |s + x|^{-2m} \, dx$$

$$\leq \int_0^{-\text{Re } s} |(-x - \text{Re } s)^{-2m} \, dx + k^{1-2m}$$

$$= \frac{1}{2m - 1} [k^{1-2m} - (-\text{Re } s)^{-2m}] + k^{1-2m}$$

$$\leq (1 + 1/(2m - 1))k^{1-2m}.$$ 

So that in all cases, if $\text{Re } s < 0$

$$\int_0^\infty |x + s|^{-2m} \, dx \leq (4m/(2m - 1))k^{1-2m},$$

so we have derived (3).

If $\text{Re } s \geq 0$, then $|s + x|^{-2m} \leq |ki + x|^{-2m}$ since

$$|s + x|^2 = (\text{Re } s + x)^2 + (\text{Im } s)^2 = 2x \text{ Re } s + x^2 + k^2 \geq |ki + x|^2.$$ 

Next, estimating as before,
The calculation of the gamma function.

\[
\int_0^\infty |ki + x|^{-2m} \, dx \leq \int_0^k k^{-2m} \, dx + \int_k^\infty x^{-2m} \, dx \leq k^{1-2m}(1 + 1/(2m - 1)),
\]

thus giving (4), and completing the proof.

On taking the exponential, we find

\[
\Gamma(s) \sim (2\pi)^{1/2} e^{-s} s^{s-1/2} \exp \left[ \sum_{k=1}^{N_1} \frac{A_{2k-1}}{s^{2k-1}} \right]
\]

(5)

where

\[
A_{2k-1} = B_{2k}/2k(2k - 1).
\]

(6)

A short calculation gives (formally)

\[
\exp \left[ \sum_{k=1}^{N_1} \frac{A_{2k-1}}{s^{2k-1}} \right] = 1 + \sum_{k=1}^{\infty} s^{-k} \sum_{\alpha \in Q(k)} \frac{A_{\alpha_1} A_{\alpha_2} \cdots A_{\alpha_n}}{j_1! j_2! \cdots j_n!}
\]

(7)

where the \(\alpha\)'s are distinct and \(Q(k)\) is the set of partitions of \(k\) into odd parts (\(\alpha_i^j\) means \(\alpha_i\) repeated \(j\) times in the partition).

Wrench \[2\] found the recurrences

\[
(2k - 1)c_{2k-1} = \frac{B_2}{2} c_{2k-2} + \frac{B_4}{4} c_{2k-4} + \cdots + \frac{B_{2k}}{2k},
\]

(8)

\[
2kc_{2k} = \frac{B_2}{2} c_{2k-1} + \frac{B_4}{4} c_{2k-3} + \cdots + \frac{B_{2k}}{2k} c_1,
\]

(9)

where \(k = 1, 2, 3, \cdots\) and \(c_0 = 1\), and these formulas are more suitable for calculating than (7).

Wrench \[2\] also gave the \(c_i\)'s for \(j = 0(1)20\), in exact form and to 50D, and also found approximations to about 6S for \(j = 21(1)30\). We give in Table 1 the exact rational values for \(j = 21(1)30\) and in Table 2 their 45D equivalents. The following corrections are necessary in Wrench's tables. In his Table 2, the last ten digits of \(c_{13}\) read 01893 93280, and should read 01894 09396. In his Table 3, entries 22, 23, 24, 26, 28, 30 can be corrected from Table 2 of this paper. Dr. Wrench confirmed the correctness of the author's value for \(c_{13}\), and that it is likely that the author's corrections to his Table 3 are also valid. It is of interest to note that while Dr. Wrench's calculations were carried out on a desk calculator, the author's were performed on a Fortran simulator of a large decimal machine (Spira \[7\]).

A further calculation revealed that entries 3, 4, 7, 8, 11, 12, 15, 16, 17 for \(c_{n+1}/c_n\) in Table XII of Spira \[3\] have errors beyond 16S. These errors did not affect the remaining tables.

Finally, we remark that estimates for the error in using

\[
\Gamma(s) \sim (2\pi)^{1/2} e^{-s} s^{s-1/2} \left\{ 1 + \frac{c_1}{s} + \frac{c_2}{s^2} + \cdots + \frac{c_k}{s^k} \right\}
\]

(10)

can be obtained from estimating

\[
\exp \left\{ \sum_{i=1}^{m} A_{2i-1}s^{-2i} + R_m \right\} - \sum_{i=1}^{k} c_i s^{-i}
\]

(11)
<table>
<thead>
<tr>
<th>$c_{21}$</th>
<th>2601 64872 18125 16297 62664 73959 14866 28167 68000 00000</th>
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<tr>
<td>$c_{22}$</td>
<td>9 09773 12459 95425 06852 27522 94225 93983 24288 04521 45053</td>
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<td></td>
<td>8 11714 40120 55050 84859 51398 75254 38279 88316 16000 00000</td>
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<td></td>
<td>15273 35577 85467 70230 23224 27280 09471 25313 62926 72693 90501</td>
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<td></td>
<td>97 40572 81446 60610 18314 16785 03052 59358 59793 92000 00000</td>
</tr>
<tr>
<td>$c_{24}$</td>
<td>2 58331 20988 61137 96374 59020 36370 49694 38721 38148 65171 20938 16393</td>
</tr>
<tr>
<td></td>
<td>1 40264 24852 83112 78663 72401 70443 95734 76381 03244 80000 00000</td>
</tr>
<tr>
<td>$c_{25}$</td>
<td>117 82196 87637 81474 07752 81743 17292 41720 16006 72563 20000 00000</td>
</tr>
<tr>
<td>$c_{26}$</td>
<td>5 18013 42908 22682 44375 77104 27952 46758 19182 33549 14089 67023 64013</td>
</tr>
<tr>
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<td>2827 72725 03307 55377 86067 61836 15018 01283 84161 41516 80000 00000</td>
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<td>$c_{27}$</td>
<td>5275 50309 09787 33965 92733 54057 99289 93424 14298 36915 19876 84094 84184 33873</td>
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<tr>
<td></td>
<td>14613 12888 42592 77641 70840 23816 85690 08674 63669 36130 51904 00000 00000</td>
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<tr>
<td>$c_{28}$</td>
<td>21148 66241 53708 11646 13223 32421 55728 12504 64870 36484 82437 46060 29560 15127</td>
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<td></td>
<td>7 01430 18644 44453 26802 00331 43209 13124 16382 56129 34264 91392 00000 00000</td>
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<tr>
<td>$c_{29}$</td>
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<td>2609 32029 35733 36615 70345 23292 73796 82188 94312 80115 46547 97824 00000 00000</td>
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<tr>
<td>$c_{30}$</td>
<td>3226 14019 20539 36286 91281 19490 56082 64758 66044 17173 68772 94520 86326 36420 80203 03641</td>
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<td>5589 16406 88340 87030 83679 48893 04472 79248 71618 02007 32705 76939 00800 00000 00000</td>
</tr>
</tbody>
</table>
and using (3) and (4), where \( m = \lceil (k + 2)/2 \rceil \). For example, for \( \text{Re} \ s \geq 0 \) and \( |s| \geq 1 \), and \( k = m = 2 \), we have

\[
\Gamma(s) = (2\pi)^{1/2} e^{-s} s^{s-1/2} \exp \left\{ \frac{1}{12s} + \frac{1}{360s^3} + R_2 \right\},
\]

where

\[
|R_2| \leq \frac{1}{90|s|^3},
\]

so

\[
|\exp R_2 - 1| \leq |R_2| \left\{ 1 + |R_2| + |R_2|^2 + \cdots \right\} \leq \frac{1}{89|s|^3}.
\]

Next,

\[
\left| \exp \left( \frac{1}{12s} + \frac{1}{360s^3} \right) - \left( 1 + \frac{1}{12s} + \frac{1}{288s^2} \right) \right|
\leq \frac{1}{360|s|^3} + \frac{1}{12 \cdot 360|s|^4} + \frac{1}{2 \cdot 360^2|s|^5} + \frac{1}{3!} \left| \frac{1}{12s} + \frac{1}{360s^3} \right|^3 + \cdots
\]

which estimates as before. Such estimates show the series for \( \Gamma(s) \) is an asymptotic series (de Bruijn [1]).

For calculations near the origin, it is best to use the functional equation \( \Gamma(s + 1) = s \Gamma(s) \) and calculate \( \Gamma(s) = \Gamma(s + j)/P(s) \), where \( P(s) \) is a polynomial. This formula could also be used for larger \( |s| \) for ultraprecise calculations where precisions are needed which are greater than the maximum precision obtainable from the asymptotic series.
totic formula. For calculations in the left half-plane with small imaginary part, one can use the equation \( \Gamma(s)\Gamma(1 - s) = \pi / \sin \pi s \).

The preparation of this paper was with the aid of NSF Grant GP-8957. I wish to thank the referee for several suggestions.

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