A Fourth-Order Finite-Difference Approximation for the Fixed Membrane Eigenproblem*

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Abstract. The fixed membrane problem \( \Delta u + \lambda u = 0 \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \), for a bounded region \( \Omega \) of the plane, is approximated by a finite-difference scheme whose matrix is monotone. By an extension of previous methods for schemes with matrices of positive type, \( O(h^4) \) convergence is shown for the approximating eigenvalues and eigenfunctions, where \( h \) is the mesh width. An application to an approximation of the forced vibration problem \( \Delta u + qu = f \) in \( \Omega \), \( u = 0 \) in \( \partial \Omega \), is also given.

1. Introduction. Let \( \Omega \) be a bounded region of the plane with smooth boundary \( \partial \Omega \). We consider the fixed membrane problem

\[
\Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial \Omega,
\]

where \( \Delta \) is the Laplacian. In [6], this problem was approximated by difference schemes which were of positive type in the interior of the region. Here, we consider a difference scheme for (1.1) which is only monotone. However, by appropriate modifications of the techniques of [6], we can prove that this scheme yields \( O(h^4) \) approximations to the eigenvalues and eigenvectors of (1.1). The principal result is Theorem 8.1. An application to a forced vibration problem is also given in Section 9.

2. The Difference Scheme. Let \( h > 0 \) be given and define the mesh \( S_h \) by

\[
\{ (ih, jh) : i, j \text{ are integers} \}.
\]

Points \( x, y \in S_h \) will be called nearest neighbors if \( |x - y| = h \), where we write

\[
|x - y| = \left( (x_1 - y_1)^2 + (x_2 - y_2)^2 \right)^{1/2}.
\]

Let \( \Omega_h^{(3)} \) be the set of points in \( S_h \cap \Omega \) having at least one nearest neighbor not in \( \Omega \). One such point might be \( x = (x_1, x_2) \) with \( (x_1 - \alpha h, x_2), (x_1, x_2 - \beta h) \in \partial \Omega \) for \( 0 < \alpha, \beta \leq 2 \). If \( (x_1 + h, x_2), (x_1, x_2 + h), (x_1, x_2 + h), (x_1, x_2 + 2h) \in \Omega \), we define

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\[ h^2 l_h(x, y) = \frac{3 - \alpha}{\alpha} + \frac{3 - \beta}{\beta}, \quad y = x, \]
\[ = -\frac{2(2 - \alpha)}{1 + \alpha}, \quad y = (x_1 + h, x_2), \]
\[ = -\frac{2(2 - \beta)}{1 + \beta}, \quad y = (x_1, x_2 + h), \]
\[ = \frac{1 - \alpha}{2 + \alpha}, \quad y = (x_1 + 2h, x_2), \]
\[ = \frac{1 - \beta}{2 + \beta}, \quad y = (x_1, x_2 + 2h), \]
\[ = 0, \quad \text{otherwise.} \]

(2.1)

Similar formulas apply at other points of \( \Omega^{(3)} \). One special case may arise, as shown in Fig. 1, where \((x_1, x_2 + h), (x_1, x_2 + 2h)\) do not lie in \( \Omega \).

In such a case \( x \) would be excluded from the difference scheme altogether and the point \((x_1 + h, x_2)\) would be added to \( \Omega^{(3)} \). For the new point, formula (2.1) would be used with \( 1 < \alpha \leq 2 \). If \( \partial \Omega \) has bounded curvature and \( h \) is sufficiently small, there will be no difficulty with the new point.

Next, let \( \Omega^{(2)} \) be those points of \( S_h \cap \Omega \), not in \( \Omega^{(3)} \) or excluded, which have a nearest neighbor in \( \Omega^{(3)} \). For \( x \in \Omega^{(3)} \) define
\[ h^2 l_h(x, y) = 4, \quad y = x, \]
\[ = -1, \quad |x - y| = h, \quad y \in S_h, \]
\[ = 0, \quad \text{otherwise.} \]

(2.2)

Finally, let \( \Omega' \) be those points of \( S_h \cap \Omega \) not in \( \Omega^{(2)} \cup \Omega^{(3)} \) or excluded. For \( x \in \Omega' \) define
\[ h^2 l_h(x, y) = 5, \quad y = x, \]
\[ = -\frac{5}{2}, \quad |x - y| = h, \quad y \in S_h, \]
\[ = \frac{1}{2}, \quad |x - y| = 2h, \quad y \in S_h, \]
\[ = 0, \quad \text{otherwise.} \]

(2.3)
Let $\Omega_h = \Omega_h^1 \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$. We approximate the Laplacian of a function $u$ vanishing on $\partial \Omega$ by
\begin{equation}
-\Delta_h u(x) = \sum_{\nu \in \partial \Omega_h} I_\nu(x, y) u(y), \quad x \in \Omega_h.
\end{equation}

Let us agree to use $C$ as a generic constant, whose value may change at each usage, but which is always independent of $h$. Then, if also $u \in C^6(\Omega)$ ($u$ has continuous sixth derivatives on the closure of $\Omega$), it can be seen from Taylor series expansions that
\begin{equation}
|u(x) - \Delta_h u(x)| \leq C h^4, \quad x \in \Omega_h,
\end{equation}
\begin{equation}
\leq C h^2, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.
\end{equation}

Bramble and Hubbard used $\Delta_h$ in [2] in approximating the Dirichlet problem for Poisson's equation.

Our difference scheme approximating (1.1) is
\begin{equation}
\Delta_h u_h(x) + \lambda_h u_h(x) = 0, \quad x \in \Omega_h.
\end{equation}

Problem (2.6) is equivalent to finding the eigenvalues and eigenvectors of the matrix $[l_\nu(x, y)]_{\nu \in \partial \Omega_h}$. In the next section, we develop some tools to use in studying this matrix which, however, have some independent interest.

3. Monotone Matrices. Let $A = (a_{ij})$ be an $n \times n$ matrix. We say $A \geq 0$ if each $a_{ij} \geq 0$ and $A \leq B$ if $B - A \geq 0$. The matrix $A$ is monotone if $Ax \geq 0$ implies $x \geq 0$ for all $x$. Thus, $A$ is monotone if and only if $A^{-1}$ exists and $A^{-1} \geq 0$. An easily recognized type of monotone matrix is a matrix of positive type. The matrix $A$ is of positive type if $A$ is indecomposable, the diagonal of $A$ is positive, the off-diagonal elements negative, and the row sums are nonnegative with at least one strictly positive. The following theorem is due to Price [8]:

**Theorem 3.1.** $A$ is monotone if and only if there exists $M$ monotone such that
\begin{enumerate}
\item $M^{-1} (M - A) \geq 0$,
\item $\rho(M^{-1}(M - A)) < 1$.
\end{enumerate}

Here $\rho$ denotes spectral radius, the maximum of the moduli of the eigenvalues. Here and in the corollaries, the "only if" part is trivial: take $M = A$. This theorem generalizes Theorem 2.7 of Bramble and Hubbard [2]. There are a number of important corollaries:

**Corollary 3.2.** $A$ is monotone if and only if there exists $M$ monotone such that
\begin{enumerate}
\item $M \geq A$,
\item $\rho(M^{-1}(M - A)) < 1$.
\end{enumerate}

**Corollary 3.3.** $A$ is monotone if and only if there exists $M$ monotone and $x > 0$ such that
\begin{enumerate}
\item $M \geq A$,
\item $Ax > 0$.
\end{enumerate}

**Proof.** By the Gerschgorin circle theorem (see [7, p. 152]),
\[ \rho(M^{-1}(M - A)) \leq \max_i [M^{-1}(M - A)x_i/x_i < 1, \]

since
\[ 0 \leq [M^{-1}(M - A)x_i = x_i - [M^{-1}Ax_i] < x_i, \]

because $Ax > 0$, $M^{-1} \geq 0$ and no row of $M^{-1}$ can be all zero.
Corollary 3.4. A is monotone if and only if there exists $M$ monotone and $x \geq 0$ such that

(i) $M \geq A$,
(ii) $Ax > 0$.

Proof. Let $\delta = \min_i |Ax_i| > 0$ and let $\epsilon = \delta/(2 \max_i |a_i|)$. Then $x + \epsilon > 0$ and $A(x + \epsilon) > 0$, so the hypotheses of Corollary 3.4 are satisfied.

Corollary 3.5. A is monotone if and only if there exist $M_1, M_2$ monotone such that

$M_1 \leq A \leq M_2$.

Proof. Let $x$ be such that $M_1 x$ is the vector with all components 1. Since $M_1$ is monotone, $x$ exists and $x \geq 0$. Also, $Ax \geq M_1 x > 0$, so the hypotheses of Corollary 3.4 are satisfied.

Corollary 3.6. A is monotone if there is $\alpha > 0$ such that $A + \alpha I$ is monotone and every eigenvalue $\lambda$ of $A$ has positive real part.

Proof. Apply Corollary 3.2. We need only show $\rho((A + \alpha I)^{-1}) < \alpha^{-1}$. But

$\rho((A + \alpha I)^{-1}) = 1/\min_{\lambda} |\alpha + \lambda|$, where $\lambda$ runs over the eigenvalues of $A$.

At this time, we also note the following:

Lemma 3.7. If the partitioned matrix

$$
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
$$

with $A$ invertible has inverse

$$
\begin{bmatrix}
W & X \\
Y & Z
\end{bmatrix}
$$

then $W - A^{-1} = -XCA^{-1}$. In particular, if $X \geq 0$, $A^{-1} \geq 0$, $C \leq 0$, then $A^{-1} \geq W$.

Proof. Since

$$
\begin{bmatrix}
W & X \\
Y & Z
\end{bmatrix}
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = 
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix},
$$

we have $WA + XC = I$. Multiply on the right by $A^{-1}$.

4. Discrete Green's Functions. The main tools in our investigations will be discrete analogues of Green's function. These are inverses of matrices related to $[h^2\delta(x, y)]_{x, y \in \Omega}$ and their nonnegativity is crucial to the investigation. This will be established, using results of the previous section.

We define then

$$
\Delta_{h, \tau} g_h(x, y) = h^{-2} \delta(x, y), \quad x \in \Omega_h \cup \Omega_h^{(2)}, \quad g_h(x, y) = \delta(x, y), \quad x \in \Omega_h^{(3)},
$$

for all $y \in \Omega_h$. This is the discrete Green's function considered by Bramble and Hubbard in [2, Eq. (4.5)]. From (4.1), we see that the matrix $[g_h(x, y)]_{x, y \in \Omega_h}$ is the inverse of the partitioned matrix

$$
M = 
\begin{bmatrix}
A & B \\
0 & I
\end{bmatrix},
$$
where \( A = [h^2l(x, y)]_{x, y \in \Omega^a \cup \Omega^b(\ast)}, B = [h^2l(x, y)]_{x, y \in \Omega^a \cup \Omega^b(\ast), y \in \Omega^b(\ast)}, \) and \( I \) is the identity on \( \Omega^a(\ast) \times \Omega^b(\ast) \). It also follows from Lemma 3.7 that the matrix \( [g_k(x, y)]_{x, y \in \Omega^a \cup \Omega^b(\ast)} \) is the inverse of \( A \). In [2], it was shown that

\[
g_k(x, y) \geq 0, \quad x, y \in \Omega^a,
\]

i.e., \( \mathcal{M} \) is monotone. Since \( g_k \) is the inverse, it follows that, for any function \( W \) defined on \( \Omega^a \), all \( x \in \Omega^a \),

\[
W(x) = h^2 \sum_{y \in \Omega^a \cup \Omega^b(\ast)} g_k(x, y)[\Delta_k W(y)] + \sum_{y \in \Omega^b(\ast)} g_k(x, y)W(y).
\]

This is analogous to Poisson's formula. In [2], the following properties were proved of \( g_k \):

\[
\sum_{y \in \Omega^a(\ast)} g_k(x, y) \leq 1, \quad \sum_{y \in \Omega^b(\ast)} g_k(x, y) \leq C, \quad h^2 \sum_{y \in \Omega^b(\ast)} g_k(x, y) \leq C,
\]

for all \( x \in \Omega^a \). Using these in (4.3), we have the inequality

\[
\max_{\Omega^a} |W| \leq C \left[ \max_{\Omega^a} |\Delta_k W| + h^2 \max_{\Omega^a(\ast)} |\Delta_k W| \right] + \max_{\Omega^a(\ast)} |W|.
\]

Now, on \( \Omega^b(\ast) \), we have

\[
W(x) = \left[ -h^2\Delta_k W(x) - h^2 \sum_{y \in \Omega^a \cup \Omega^b(\ast)} l_k(x, y)W(y) \right] / h^2l_k(x, x),
\]

and from this and (2.1), we see that

\[
\max_{\Omega^b(\ast)} |W| \leq Ch^2 \max_{\Omega^a(\ast)} |\Delta_k W| + \theta \max_{\Omega^a} |W|,
\]

where

\[
\theta = \max_{y \in \Omega^a(\ast)} \sum_{x \in \Omega^a : y \neq x} |l_k(x, y)|/l_k(x, x) < 1.
\]

Putting (4.8) into (4.7) and rearranging, we have

\[
\max_{\Omega^a} |W| \leq C \left[ \max_{\Omega^a} |\Delta_k W| + h^2 \max_{\Omega^a(\ast)} |\Delta_k W| \right].
\]

Let us now use (4.7) to estimate \( W = \Phi_k - \varphi \) where \( \varphi \) is the torsion function defined by \( \Delta \varphi = -1 \) on \( \Omega, \varphi = 0 \) on \( \partial \Omega \) and \( \Phi_k(x) = h^2 \sum_{y \in \Omega^b} g_k(x, y) \), which satisfies \( \Delta_k \Phi_k = -1 \) on \( \Omega^a \cup \Omega^b(\ast) \). If \( \partial \Omega \) is sufficiently smooth, \( \varphi \) satisfies (2.5) and we see from (4.7) that

\[
\max_{\Omega^a} |\Phi_k - \varphi| \leq Ch^4 + \max_{\Omega^a(\ast)} |\Phi_k - \varphi| \leq Ch^4 + \max_{\Omega^a(\ast)} |\Phi_k| + \max_{\Omega^a(\ast)} |\varphi|.
\]

Now, \( \varphi = 0 \) on \( \partial \Omega \), so \( |\varphi(x)| \leq C \max_{\varphi \in \partial \Omega} |x - \partial \Omega| = \min_{\varphi \in \partial \Omega} |x - y| \leq Ch \). Also, \( \Phi_k = h^2 \) on \( \Omega^b(\ast) \) by definition. Hence,

\[
|\Phi_k(x)| \leq |\varphi(x)| + \max_{\Omega^a} |\Phi_k - \varphi| \leq Ch.
\]
for \(|x - \partial \Omega| \leq Ch\), i.e.,
\begin{equation}
(4.10) \quad h^2 \sum_{y \in \Omega_h} g_h(x, y) \leq Ch.
\end{equation}

Next, we consider the function
\[ f_h(x, y) = C_1 - C_2 \log (|x - y|^2 + h^2). \]

It is easily verified that
\[ \Delta_{h, z} f_h(x, y) \geq 0, \quad x \in \Omega_h' \cup \Omega_h^{(2)}, \quad y \neq x, \]
\[ \Delta_{h, z} f_h(x, y) \geq h^2, \quad x \in \Omega_h' \cup \Omega_h^{(2)}, \quad y = x, \]
provided \( C_2 \geq \frac{1}{2} \log 2 \). If we choose
\[ C_1 = C_2 \max_{x, y \in \Omega_h} \log (|x - y|^2 + h^2), \]
then \( f_h(x, y) \geq 0 \) for \( x, y \in \Omega \). Thus, we see that
\[ \mathcal{M}(f_h - g_h) \geq 0, \]
and, since \( \mathcal{M} \) is monotone,
\begin{equation}
(4.11) \quad 0 \leq g_h(x, y) \leq C_1 - C_2 \log (|x - y|^2 + h^2).
\end{equation}

Analogous inequalities to (4.11) are proved by Bramble and Thomée in [3] for discrete Green's functions of positive-type operators. Here, we see monotonicity was sufficient.

An easy consequence of (4.11) is
\begin{equation}
(4.12) \quad h^2 \sum_{y \in \Omega_h} [g_h(x, y)]^2 \leq C.
\end{equation}

5. More Inequalities for Green's Functions. This section will be devoted to derivations of some inequalities of more difficulty than those of the previous section.

Recall that \( \mathcal{O} = [g_h(x, y)]_{x, y \in \Omega_h} \) is the inverse of \( [h^2 f_h(x, y)]_{x, y \in \Omega_h} \).

The inequality which we next wish to derive is
\begin{equation}
(5.1) \quad \sum_{x \in \Omega_h} g_h(x, y) \leq C
\end{equation}
for all \( x \in \Omega_h \), where \( \Omega_h' = \{ x \in \Omega_h' : f_h(x, y) \neq 0 \text{ for some } y \in \Omega_h^{(2)} \cup \Omega_h^{(3)} \} \). The method of proof is the matrix splitting technique employed by Bramble and Hubbard in [2]. The analysis which follows is regrettably detailed.

Let us write
\begin{equation}
(5.2) \quad \mathcal{O} = [I - H_1 - H_2]^{-1} \tilde{D}^{-1},
\end{equation}
where \( \tilde{D} \) is the diagonal matrix with
\[ \tilde{d}_{xx}^{-1} = 1, \quad x \in \Omega_h^{(3)}, \]
\[ = \frac{1}{4}, \quad x \in \Omega_h^{(2)}, \]
\[ = \frac{1}{5}, \quad x \in \Omega_h'. \]
and

\[
[H_1]_{xy} = \begin{cases} 
\frac{2}{h}, & x \in \Omega_h', \quad |x - y| = h, \\
\frac{1}{h}, & x \in \Omega_h^{(2)}, \quad |x - y| = h, \\
0, & \text{otherwise},
\end{cases}
\]

\[
[H_2]_{xy} = \begin{cases} 
\frac{2}{h}, & x \in \Omega_h', \quad |x - y| = h, \\
-\frac{1}{h}, & x \in \Omega_h', \quad |x - y| = 2h, \\
\frac{1}{h}, & x \in \Omega_h^{(2)}, \quad |x - y| = h, \\
0, & \text{otherwise}.
\end{cases}
\]

Let us define the diagonal matrix \( D \) by

\[
(D_{xx})^{-1} = \sum_{y \in \Omega_h} (I - H_1)_{xy} = \frac{2}{h}, \quad x \in \Omega_h', \\
= \frac{1}{h}, \quad x \in \Omega_h^{(2)}, \\
= 1, \quad x \in \Omega_h^{(3)},
\]

so that \( D(I - H_1) \) has row sums one, i.e.,

\[
\sum_{y \in \Omega_h} [D(I - H_1)]_{xy} = \sum_{y \in \Omega_h} [(I - H_1)^{-1} D^{-1}]_{xy} = 1.
\] (5.3)

We write \([I - H_1 - H_2] = [D^{-1}(I - H)] [D(I - H_1)], \) where \( H = DH_2(I - H_1)^{-1}D^{-1}. \)

Thus, by (5.3),

\[
\sum_{y \in \Omega_h} [D^{-1}(I - H)]_{xy} = \sum_{y \in \Omega_h} [(I - H_1)_{xy} D(I - H_1)]_{xy} \\
= \sum_{x \in \Omega_h} [(I - H_1 - H_2)_{xx} = 0, \quad x \in \Omega_h' \cup \Omega_h^{(2)}, \\
= 1, \quad x \in \Omega_h^{(3)}.
\] (5.4)

Now, we consider the characteristic function of \( \Omega_h' \):

\[
\chi(x) = \begin{cases} 
1, & x \in \Omega_h', \\
0, & \text{otherwise},
\end{cases}
\]

Then

\[
1 \geq \chi(x) = \left\{ [(I - H)^{-1} D][D^{-1}(I - H)\chi]\right\}_x \\
= \sum_{y \in \Omega_h} [(I - H)^{-1} D]_{xy} [D^{-1}(I - H)\chi]_y \\
+ \sum_{y \in \Omega_h^{(2)}} [(I - H)^{-1} D]_{xy} [D^{-1}(I - H)\chi]_y \\
= \sum_{x \in \Omega_h} [(I - H)^{-1} D]_{xy} \sum_{x \in \Omega_h} [D^{-1}(I - H)]_{yx} \\
- \sum_{x \in \Omega_h} [(I - H)^{-1} D]_{xy} [D^{-1}(I - H)(1 - \chi)]_y \\
+ \sum_{x \in \Omega_h^{(2)}} [(I - H)^{-1} D]_{xy} [D^{-1}(I - H)\chi]_y.
\]
By (5.4), the first term vanishes. Using the definitions of $H$ and $x$, this can be written as

$$
\sum_{y \in \Omega_2} [(I - H)^{-1} D]_{xy} \sum_{y \in \Omega_2 \cup \Omega_3} [H_2 (I - H_1)^{-1} D^{-1}]_{xy}
$$

\begin{equation}
5.5
\end{equation}

$$
- \sum_{y \in \Omega_2 \cup \Omega_3} [(I - H)^{-1} D]_{xy} \sum_{y \in \Omega_2} [H_2 (I - H_1)^{-1} D^{-1}]_{xy} \leq 1.
$$

Now, we estimate the factors in each term. First, note that $(I - H)^{-1} \geq 0$. This is not obvious, but follows from $H \geq 0$ and $\rho(H) < 1$. That $H \geq 0$, is due to $0 \leq H_2 (I - H_1)^{-1} = H_2 + H_2 H_1 + \cdots$, since the negative terms in $H_2$ are cancelled by positive terms in $H_2 H_1$ as in [2]. That $\rho(H) < 1$ is due to $\rho(H) = \rho((I - H_1)^{-1} H_2) < 1$, since the row sums of

$$(I - (I - H_1)^{-1} H_2) = (I - H_1)^{-1} (I - H_1 - H_2)
$$

$$
= (I - H_1 - H_2) + H_1 (I - H_1 - H_2) + \cdots
$$

are positive. Again negative row sums of $(I - H_1 - H_2)$ are cancelled by corresponding positive row sums of $H_1 (I - H_1 - H_2)$.

Next, for $y \in \Omega_2 \cup \Omega_3$,

$$
\sum_{y \in \Omega_2} [H_2 (I - H_1)^{-1} D^{-1}]_{xy}
$$

\begin{equation}
5.6
\end{equation}

$$
\leq \sum_{y \in \Omega_2} [H_2 (I - H_1)^{-1} D^{-1}]_{xy} = \sum_{y \in \Omega_2} [D^{-1} - D^{-1} (I - H)]_{xy}
$$

$$
\leq 1 - \sum_{y \in \Omega_2} [D^{-1} (I - H)]_{xy} = 1 - \sum_{y \in \Omega_2} [I - H_1 - H_2]_{xy} \leq 1.
$$

Now, we consider, for $y \in \Omega''$, the term

$$
\sum_{x \in \Omega_2 \cup \Omega_3} [H_2 (I - H_1)^{-1} D^{-1}]_{xy}.
$$

Expanding the summand in a Neumann series, it becomes

$$
[(H_2 + H_2 H_1 + H_2 H_1^2 + \cdots) D^{-1}]_{xy}.
$$

If $y \in \Omega''$, $z \in \Omega_2 \cup \Omega_3$ is such that $|y - z| = 2h$, then $[H_2]_{xy} = -1/60$. However, let $x$ be the point such that $|y - x| = |x - z| = h$. Then $[H_2 H_1]_{xy}$ contains the term $[H_2]_{xy} [H_1]_{y'z} = 4/225$. Similarly, each negative term in $H_2 H_1$ is compensated for by a positive term in $H_2 H_1^2$. Thus, for $y \in \Omega''$,

$$
\sum_{x \in \Omega_2 \cup \Omega_3} [H_2 (I - H_1)^{-1} D^{-1}]_{xy} \geq \left[-\frac{1}{60} + \frac{4}{225}\right] \frac{1}{2} = \frac{1}{1800}.
$$

It follows from (5.5) and the above that

$$
\sum_{y \in \Omega''} [(I - H)^{-1} D]_{xy} \leq 1800 \left\{1 + \sum_{y \in \Omega_2 \cup \Omega_3} [(I - H)^{-1} D]_{xy}\right\}.
$$

By similar reasoning, using the function

$$
\chi(x) = 1, \quad x \in \Omega'' \cup \Omega_3,
$$

$$
= 0, \quad x \in \Omega_3,
$$

it can be shown that $\sum_{x \in \Omega''} [(I - H)^{-1} D]_{xy} \leq C$. The argument is carried out in [2, Lemma 3.3]. Finally, we note from (5.4) that
(5.8) \[ 1 = \sum_{y \in \Omega_h} [(I - H)^{-1}D]_{xy} \sum_{x \in \Omega_h} [D^{-1}(I - H)]_{xy} = \sum_{y \in \Omega_h} [(I - H)^{-1}D]_{xy}. \]

Combining the above with (5.7), we see that

(5.9) \[ \sum_{y \in \Omega_h} [(I - H)^{-1}D]_{xy} \leq C. \]

From (5.2) and (5.3), we finally have

\[ \sum_{y \in \Omega_h} g_h(x, y) = \sum_{y \in \Omega_h} [(I - H_1 - H_2)^{-1}D^{-1}]_{xy} = \frac{1}{h} \sum_{y \in \Omega_h} [(I - H_1 - H_2)^{-1}]_{xy} \]

or, from (5.9),

(5.10) \[ \sum_{y \in \Omega_h} g_h(x, y) \leq C, \]

the desired estimate.

We next define another Green's function \( G_h \) by

(5.11) \[ -\Delta_h G_h(x, y) = h^{-2} \delta(x, y), \quad x, y \in \Omega_h. \]

Although \( G_h \) may not be nonnegative, it is a perturbation of \( g_h \). We have

**Theorem 5.1.** For any mesh function \( S \),

\[ \max_{x \in \Omega_h} \sum_{y \in \Omega_h} |G_h(x, y) - g_h(x, y)|S(y)| \leq C \max_{x \in \Omega_h} |S| + \max_{y \in \Omega_h} \sum_{x \in \Omega_h} g_h(x, y) |S(y)| \]

**Proof.** Let \( x_0 \in \Omega \) be the point where \( \sum_{y \in \Omega_h} |G_h(x, y) - g_h(x, y)|S(y)| \) attains its maximum and let

\[ W(x) = \sum_{y \in \Omega_h} [G_h(x, y) - g_h(x, y)]S^*(y), \]

where \( S^*(y) = |S(y)| \sgn [G_h(x_0, y) - g_h(x_0, y)] \). Employing (4.9), we have

\[ \max_{\Omega_h} |W| \leq C \max_{\Omega_h} |h^2 \Delta_h W| \]

\[ \leq C \left[ \max_{\Omega_h} |S| + \max_{y \in \Omega_h} \sum_{x \in \Omega_h} g_h(x, y) |S^*(y)| \right], \]

and (5.12) follows.

**Corollary 5.2.** For all \( x, z \in \Omega_h \),

(5.13) \[ \sum_{y \in \Omega_h} |G_h(x, y)| \leq C, \]

(5.14) \[ h^2 \sum_{y \in \Omega_h} |G_h(x, y)| \leq C, \]

(5.15) \[ |G_h(x, z)| \leq C |\log h|, \]

(5.16) \[ h^2 \sum_{y \in \Omega_h} |G_h(x, y)|^2 \leq C, \]
and for $|x - \partial \Omega| \leq Ch$,

$$h^2 \sum_{y \in \partial \Omega} |G_h(x, y)| \leq Ch.$$  

Proof. For (5.13), employ the characteristic function of $\Omega_h^{(1)} \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$ as $S$ in (5.13). Then apply the triangle inequality and (4.4), (4.5), and (5.10). For (5.14), let $S = h^2$ and use (4.6) and (4.10), respectively. For (5.15), let $S(y) = \delta(y, z)$ in (5.12), apply the triangle inequality and (4.11). For (5.16), let $x_0$ be the point where $\max_{x \in \Omega} h^2 \sum_{y \in \Omega} |G_h(x, y)|^2$ is attained, and let $S(y) = h^2 G_h(x_0, y)$ in (5.12), from which it follows that

$$h^2 \sum_{y \in \Omega} |G_h(x, y)|^2 \leq Ch^2 \max_{y \in \Omega^{(1)}(x)} |G_h(x_0, y)| + \max_{x \in \Omega} h^2 \sum_{y \in \Omega} g_h(x, y)G_h(x_0, y).$$

Again, using (5.12) with $S(y) = h^2 g_h(x, y)$ for $x$ fixed,

$$h^2 \sum_{y \in \Omega} G_h(x_0, y)g_h(x, y) \leq Ch^2 \max_{y \in \Omega^{(1)}(x)} |g_h(x, y)| + \max_{x \in \Omega} h^2 \sum_{y \in \Omega} g_h(x_0, y)g_h(x, y).$$

By (4.11), this term can be seen to be bounded. Finally, letting $S(y) = h^2 \delta(y_0, y)$ in (5.12), we have, for any $y_0 \in \Omega$,.

$$|h^2 G_h(x_0, y_0)| \leq C \left[ h^2 + \max_{x \in \Omega} h^2 g_h(x, y_0) \right],$$

which indeed tends to zero as $h$ does, by (4.11), and (5.16) follows. For (5.17) use $S = h^2$ and (4.10).

We require yet one more Green's function $G_h'$ defined by

$$-\Delta_h G_h'(x, y) = h^{-2} \delta(x, y), \quad x \in \Omega_h', \quad G_h'(x, y) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)},$$

for all $y \in \Omega_h$. Thus, the matrix $[G_h'(x, y)]_{x, y \in \Omega_h'}$ is the inverse of the symmetric matrix $\mathcal{J} = [h^2 \delta_h(x, y)]_{x, y \in \Omega_h'}$. We show $\mathcal{J}$ is monotone by applying Corollary 3.6. First, we show $\mathcal{J}$ is monotone from Corollary 3.5: we define $M_i$ by

$$[M_1]_{x, y} = \begin{cases} \frac{16}{3}, & x = y, \\ -\frac{4}{3}, & |x - y| = h, \\ 0, & \text{otherwise}, \end{cases}$$

for $x, y \in \Omega_h'$, and we define

$$[M_2]_{x, y} = \begin{cases} \frac{8}{\sqrt{12}}, & x = y, \\ -\frac{1}{\sqrt{12}}, & |x - y| = h, \\ 0, & \text{otherwise}. \end{cases}$$

Since $M_1$ and $M_2$ are of positive type, they are monotone, hence, so is $M_2$, and it is easy to see that

$$M_1 \leq \mathcal{J} \leq M_2.$$
Thus, $\xi$ is monotone if its eigenvalues, necessarily real by symmetry, are positive. But these are $h^2\mu_k^{(i)}$, where $\mu_k^{(i)}$ is the $i$th eigenvalue satisfying

$$\Delta_k V_k^{(i)}(x) + \mu_k^{(i)} V_k^{(i)}(x) = 0, \ x \in \Omega_k^*, \ V_k^{(i)}(x) = 0, \ x \in \Omega_k^{(2)} \cup \Omega_k^{(3)}.$$  

In the next section, we shall show that indeed $|\mu_k^{(i)} - \lambda^{(i)}| \to 0$ as $h \to 0$, for $\lambda^{(i)}$ the $i$th eigenvalue of (1.1), which is strictly positive. Thus, for $h$ sufficiently small, $\xi$ is monotone and $G'_k$ nonnegative. Thus, as a consequence of Lemma 3.7,

$$0 \leq G'_k(x, y) \leq g_k(x, y), \ x, y \in \Omega_k.$$  

From (5.20), we see that all of the inequalities proved for $g_k$ hold for $G'_k$. In particular, the difficult inequality (5.10) does, from which we prove the key inequality

$$\max_{\Omega_k} |W| \leq C \left[ \max_{\Omega_k} |\Delta_k W| + \max_{\Omega_k^{(*)} \cup \Omega_k^{(*)'}} |W| \right],$$  

for all $W$ defined on $\Omega_k$. To prove this, let

$$W^*(x) = W(x), \ x \in \Omega_k^*,$$

$$= 0, \ x \in \Omega_k^{(2)} \cup \Omega_k^{(3)}.$$  

Then, by (5.18),

$$W^*(x) = h^2 \sum_{y \in \Omega_k^*} G'_k(x, y)[-\Delta_k W^*(y)]$$

$$= h^2 \sum_{y \in \Omega_k^*} G'_k(x, y)[-\Delta_k W(y)] + h^2 \sum_{y \in \Omega_k^*} G'_k(x, y)[\Delta_k W(y) - \Delta_k W^*(y)],$$  

and (5.21) follows from (4.6), (5.10), and (5.20).

6. Convergence of $\mu_k^{(n)}$ to $\lambda^{(n)}$. In this section, we show that the eigenvalue $\mu_k^{(n)}$ of

$$\Delta_k V_k^{(n)}(x) + \mu_k^{(n)} V_k^{(n)}(x) = 0, \ x \in \Omega_k^*, \ V_k^{(n)}(x) = 0, \ x \in \Omega_k^{(2)} \cup \Omega_k^{(3)},$$  

converges to $\lambda^{(n)}$ of (1.1) for each $n$. We will use the variational principles associated with (1.1) and (6.1), and a technique of Weinberger [9].

The $n$th eigenvalue of (1.1) can be characterized by

$$\lambda^{(n)} = \min \max \ D(u) / \int_\Omega u^2 \ dx,$$

where $u = \alpha_1 u_1 + \cdots + \alpha_n u_n$, the max is with respect to the scalars $\alpha_1, \ldots, \alpha_n$, the min is with respect to choices of linearly independent $u_1, \ldots, u_n$, continuous, piecewise differentiable functions vanishing on $\partial \Omega$, and $D(u)$ is the Dirichlet integral.

Similarly, the $n$th eigenvalue of (1.1) can be characterized by

$$h^2 \sum_{x \in \Omega_k^*} \left[ \frac{U_{z_1}^2}{2} + U_{z_2}^2 + \frac{h^2}{12} U_{z_1, z_2}^2 + \frac{h^2}{12} U_{z_1, z_2}^2 \right],$$

where $U = \alpha_1 U_1 + \cdots + \alpha_n U_n$, the max is with respect to the scalars $\alpha_1, \ldots, \alpha_n$, the min is with respect to choices of linearly independent mesh functions $U_1, \ldots, U_n$ vanishing on $\Omega_k^{(2)} \cup \Omega_k^{(3)}$, the sum is over the mesh points of $\Omega_k^*$, and subscript $x, (\xi_i)$
denotes forward (backward) difference quotient in the $x_i$ direction, $i = 1, 2$, i.e., $U_{x_i}(y_1, y_2) = [U(y_1 + h, y_2) - U(y_1, y_2)]/h$, etc.

First, we show

\begin{equation}
\mu_h^{(n)} \leq \lambda^{(n)} + O(h).
\end{equation}

Let $u^{(1)}, \ldots, u^{(n)}$ be eigenfunctions associated with $\lambda^{(1)}, \ldots, \lambda^{(n)}$ in (1.1), $u = \alpha_1 u^{(1)} + \cdots + \alpha_n u^{(n)}$, and define

\begin{align*}
u(x) &= h^{-1} \int_{Q_h(x)} u(y)\,dy, \quad x \in \Omega', \\
&= 0, \quad x \in \Omega_h(2) \cup \Omega_h(3),
\end{align*}

where $Q_h(x) = \{(y_1, y_2) : |x_1 - y_1| \leq \frac{1}{2}h, |x_2 - y_2| \leq \frac{1}{2}h\}$ is the square of side $h$ centered at $x$. Put this $U$ in (6.3). Employing inequalities (2.14), (2.22) and (8.6) of Weinberger [9], we see that

\begin{equation*}
p_k \leq \max \left\{ \frac{\int_a u^2 \,dx - (\frac{h^2}{\pi^2})D(u)}{\int_a u^2 \,dx} \right\},
\end{equation*}

and Hubbard [5, pp. 568–569], has shown

\begin{equation*}
\int_a \left\{ \left( \frac{\partial^3 u}{\partial x_1^3} \right)^2 + \left( \frac{\partial^3 u}{\partial x_2^3} \right)^2 \right\} \,dx \leq C\lambda^{(n)}u^2.
\end{equation*}

From these, (6.4) follows.

Next, we show

\begin{equation}
\lambda^{(n)} \leq \mu_h^{(n)} + O(h).
\end{equation}

Let $V^{(1)}_h, \ldots, V^{(n)}_h$ be eigenvectors associated with $\mu_h^{(1)}, \ldots, \mu_h^{(n)}$ in (6.1), $U = \alpha_1 V^{(1)}_h + \cdots + \alpha_n V^{(n)}_h$, and define $u$ to be the continuous, piecewise linear function interpolating $U$ (see [9, Section 6]). Then, by (6.4), (6.7) of [9] we see that

\begin{align*}
\lambda^{(n)} &\leq \max \frac{h^2}{\alpha^2} \sum \left( U_{x_1}^2 + U_{x_2}^2 \right) \\
&\leq \max \frac{h^2}{\alpha^2} \sum \left[ U_{x_1}^2 + U_{x_2}^2 + \frac{h^2}{12} U_{x_1}^2 + \frac{h^2}{12} U_{x_2}^2 \right] \\
&= \frac{\mu_h^{(n)}}{1 - \frac{1}{h^2}\mu_h^{(n)}}
\end{align*}

and we obtain (6.5). Combining (6.4) and (6.5), we have

\begin{equation}
|\mu_h^{(n)} - \lambda^{(n)}| \to 0 \quad \text{as} \quad h \to 0,
\end{equation}

for each $n = 1, 2, \cdots$. 
7. Convergence of $\lambda^{(n)}_k$ to $\lambda^{(n)}$ by Perturbation. Next, we will show that
the $\lambda^{(n)}_k$ are a perturbation of the $\mu^{(n)}_k$, and that as $h$ tends to zero, $\lambda^{(n)}_k$ tends to $\mu^{(n)}_k$,
hence to $\lambda^{(n)}$, by Section 6. We employ the following theorem of Wielandt:

**Theorem 7.1.** If $A$, $B$ are $v \times v$ matrices and $A$ has an orthonormal basis of eigenvectors,
then the eigenvalues of $B$ lie in the union of the $v$ discs $|\mu^{(i)} - z| \leq ||A - B||_2$,
where the $\mu^{(i)}$ are the eigenvalues of $A$. If $k$ discs are disjoint from the others, they
contain exactly $k$ eigenvalues of $B$.

In the theorem, $|| \cdot ||_2$ is the spectral norm of a matrix, defined by

$$||M||_2 = \sup_\xi ||M\xi||_2/||\xi||_2,$$

where $||\xi||_2 = \left(\sum_{i=1}^n |\xi_i|^2\right)^{1/2}$

for a $v$-vector $\xi = (\xi_1, \cdots, \xi_v)$. For a proof of the theorem, see [6].

We apply the theorem as follows. For $A$, we take the matrix $[h^2 G_k(x, y)]_{x, y}$. Note that the minor $[h^2 G_k(x, y)]_{x, y}$ is symmetric, while $h^2 G_k(x, y) = 0$ for $x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}$, so that $A$ has an orthonormal basis of eigenvectors, and the eigenvalues
are simply $|\mu^{(i)}|^{-1}$ plus some zeros. For $B$, we take the matrix $[h^2 G_k(x, y)]$ whose
eigenvalues are $|\lambda^{(i)}|^{-1}$. Thus, we must estimate $||h^2(G_k - G'_k)||_2$. However, for any matrix,

$$||M||_2 \leq [\rho(MM^T)]^{1/2} \leq ||MM^T||^{1/2},$$

where $|| \cdot ||_1$ is the maximum of the absolute row sums of the matrix. This is a consequence of the Gerschgorin circle theorem (see, e.g., [7, p. 146]). Thus, we need to estimate

$$h^4 \max_{x, y, z} \sum_{x, y, z} |G_k(x, y, z) - G'_k(x, y, z)|G_k(y, z) - G'_k(y, z)|.$$

Let $x_0$ be the point where the max is attained and put

$$\sigma(y) = \text{sgn} \sum_{x, y, z} [G_k(x_0, y, z) - G'_k(x_0, y, z)][G_k(y, z) - G'_k(y, z)].$$

Then, let

$$W(x) = h^4 \sum_{y, z} [G_k(x, y, z) - G'_k(x, y, z)][G_k(y, z) - G'_k(y, z)]\sigma(y)$$

in (4.9). Then, (7.1) is bounded by

$$Ch^4 \max_{x, y, z} \sum_{y, z} |G_k(y, z) - G'_k(y, z)|$$

in (4.9). Then, (7.1) is bounded by

$$Ch^4 \max_{x, y, z} \sum_{y, z} |G_k(y, z) - G'_k(y, z)|$$

Now,

$$h^2 \sum_{y, z} |G_k(y, z) - G'_k(y, z)| \leq C \max_{x, y, z} |G_k(y, z) - G'_k(y, z)| \leq C|h|,$$

by (4.11), (5.15) and (5.20). Using this in (7.2) and also (4.10) and (5.20), we have
(7.2) bounded by $Ch^4 |\log h|$, which tends to zero as $h$ tends to zero. Thus, the radii
of the discs in Theorem 7.1 tend to zero as $h$ does. Since the $\mu^{(n)}_k$ tend to the $\lambda^{(n)}$,
which have no finite accumulation point, the disc associated with $|\mu^{(n)}_k|^{-1}$ for any
fixed \( n \) eventually becomes disjoint from the remaining discs. Consequently, for any fixed \( n \) and \( \varepsilon > 0 \), there is \( h \) sufficiently small that

\[
|\lambda_h^{(n)} - \lambda^{(n)}| < \varepsilon.
\]

8. Main Theorem. We are now ready to state and prove our main theorem:

**Theorem 8.1.** Let \( \lambda^{(n)} \) be the \( n \)-th eigenvalue of (1.1), let \( \lambda_h^{(n)} \) be the \( n \)-th eigenvalue of (2.6) with associated eigenvector \( U_h^{(n)} \). For each \( n = 1, 2, \cdots \), there are constants \( C_n, h_n \) such that for \( h < h_n \)

\[
|\lambda_h^{(n)} - \lambda^{(n)}| < C_n h^4,
\]

and there is an eigenfunction \( u_h^{(n)} \) associated with \( \lambda_h^{(n)} \) such that

\[
\max_{\Omega_h} |U_h^{(n)} - u_h^{(n)}| < C_n h^4.
\]

**Proof.** With the machinery generated in the previous sections, our proof will have exactly the form of the proof of the corresponding Theorem 5.1 of [6]. For this reason, we only sketch the proof.

By (7.3)

\[
|\lambda_h^{(n)}| \leq C_n.
\]

By (5.11), (2.6) is equivalent to

\[
U_h^{(n)}(x) = \lambda_h^{(n)} h^2 \sum_{y \in \Omega_h} G_h(x, y) U_h^{(n)}(y), \quad x \in \Omega_h.
\]

Let us use the notations

\[
\langle U, V \rangle_h = h^2 \sum_{y \in \Omega_h} U(y) \overline{V(y)}, \quad ||U||_h = \langle U, U \rangle_h^{1/2},
\]

\[
\langle U, V \rangle_h' = h^2 \sum_{y \in \Omega_h} U(y) \overline{V(y)}, \quad ||U||_h' = \langle U, U \rangle_h'^{1/2}.
\]

If \( U_h^{(n)} \) is normalized by requiring \( ||U_h^{(n)}||_h = 1 \), then (8.4), (8.3), the Schwarz inequality, and (5.16) show

\[
\max_{\Omega_h} |U_h^{(n)}| \leq C_n.
\]

From (8.4), (8.5) and (5.17), we see that for \( |x - \partial \Omega| \leq Ch \)

\[
|U_h^{(n)}(x)| \leq C_n h.
\]

Let us suppose that \( \lambda^{(n)} = \lambda^{(n+1)} = \cdots = \lambda^{(n+m)} \) is an eigenvalue of multiplicity \( m + 1 \). Since \( \Delta_h \) restricted to \( \Omega_h' \) is symmetric, the eigenvectors \( V_h^{(n)} \) of (6.1) are a complete orthonormal basis on \( \Omega_h' \):

\[
\langle V_h^{(i)}, V_h^{(j)} \rangle_h' = \delta(i, j).
\]

If we set

\[
\rho_h^{(i)} = \sum_{j=n}^{n+m} \langle U_h^{(i)}, V_h^{(j)} \rangle \overline{V_h^{(j)}}, \quad i = n, \cdots, n + m,
\]

then
(8.7) \[ ||U_h^{(i)} - \tilde{U}_h^{(i)}||_k^2 \leq C_nh, \quad i = n, \ldots, n + m.\]

This follows from Parseval's identity:

\[||U_h^{(i)}||_k^2 = \langle U_h^{(i)}, \tilde{V}_h^{(i)} \rangle_k + \sum_{i=n, \ldots, n+n+m} |\langle U_h^{(i)}, V_h^{(i)} \rangle_k|^2\]

\[= \langle U_h^{(i)}, \tilde{V}_h^{(i)} \rangle_k + \sum_{i=n, \ldots, n+n+m} \left| \frac{\mu_i^{(i)}}{\mu_i^{(i)} - \lambda_i^{(i)}} \langle H_h^{(i)}, V_h^{(i)} \rangle_k \right|^2,\]

where \(H_h^{(i)}\) is uniquely defined by

\[\Delta_h H_h^{(i)}(x) = 0, \quad x \in \Omega_1', \quad H_h^{(i)}(x) = U_h^{(i)}(x), \quad x \in \Omega_2' \cup \Omega_3'.\]

It follows from our hard-won inequality (5.21) that

\[\max_{\Omega_h} |H_h^{(i)}| \leq \max_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)}| \leq C_nh,\]

by (8.6), and so

\[||U_h^{(i)} - \tilde{V}_h^{(i)}||_k^2 = ||U_h^{(i)}||_k^2 - \langle U_h^{(i)}, \tilde{V}_h^{(i)} \rangle_k \leq C_nh^2.\]

In a very similar manner, we show that if

\[\tilde{V}_h^{(i)} = \sum_{i=n}^{n+m} \langle U_h^{(i)}, V_h^{(i)} \rangle_k V_h^{(i)}, \quad i = n, \ldots, n + m,\]

then

(8.8) \[||u^{(i)} - \tilde{v}_h^{(i)}||_k^2 \leq C_nh, \quad i = n, \ldots, n + m.\]

From (8.8), we can conclude that the \((m + 1) \times (m + 1)\) matrix \([\langle u^{(i)}, V_h^{(i)} \rangle_k]\), \(i, j = n, \ldots, n + m\), is nonsingular. In particular then, there are eigenvectors

\[u_h^{(i)} = \sum_{i=n}^{n+m} a_{ij}(h)u^{(i)}, \quad i = n, \ldots, n + m,\]

in the eigenmanifold associated with \(\lambda^{(i)}\) such that

(8.9) \[\langle u_h^{(i)}, V_h^{(i)} \rangle_k = \langle U_h^{(i)}, V_h^{(i)} \rangle_k, \quad i, j = n, \ldots, n + m.\]

Moreover, the coefficients \(a_{ij}(h)\) are bounded independently of \(h\).

Then, it follows from (8.9) and Parseval's identity that

\[||U_h^{(i)} - u_h^{(i)}||_k^2 = h^2 \sum_{\Omega_h^{(i)} \cup \Omega_h^{(i) \dagger}} \left| U_h^{(i)} - u_h^{(i)} \right|^2 + \sum_{i=n, \ldots, n+n+m} \left| \langle U_h^{(i)} - u_h^{(i)}, V_h^{(i)} \rangle_k \right|^2\]

\[= h^2 \sum_{\Omega_h^{(i)} \cup \Omega_h^{(i) \dagger}} |U_h^{(i)} - u_h^{(i)}|^2\]

\[+ \sum_{i=n, \ldots, n+n+m} \left| \frac{\mu_i^{(i)}}{\mu_i^{(i)} - \lambda_i^{(i)}} \langle H_h^{(i)}, V_h^{(i)} \rangle_k - \frac{\mu_i^{(i)}}{\mu_i^{(i)} - \lambda_i^{(i)}} \langle \tilde{H}_h^{(i)}, V_h^{(i)} \rangle_k \right|^2,\]

where \(\tilde{H}_h^{(i)}\) is defined by

\[\Delta_h \tilde{H}_h^{(i)}(x) = 0, \quad x \in \Omega_1', \quad \tilde{H}_h^{(i)}(x) = u_h^{(i)}(x), \quad x \in \Omega_2' \cup \Omega_3'.\]

Since \(|u_h^{(i)}(x)| \leq C_nh\) for \(|x - \partial\Omega| \leq C_nh\), we see that

(8.10) \[||U_h^{(i)} - u_h^{(i)}||_k \leq C_nh.\]
From (8.10), we also have

\[(8.11) \quad |(U_h^{(i)}, u_h^{(i)})| \geq 1 - C_i h^2.\]

Inequality (8.11) is the key inequality needed to prove the first half of Theorem 8.1, for now

\[(8.12) \quad \langle U_h^{(i)}, \Delta u_h^{(i)} - \Delta_x u_h^{(i)} \rangle = \langle \tau_h u_h^{(i)}, u_h^{(i)} \rangle + \langle \tau_h u_h^{(i)}, \Delta u_h^{(i)} - \Delta_x u_h^{(i)} \rangle,
\]

obtained by adding and subtracting terms. We have used the notations

\[\tau_h u_h^{(i)} = \Delta u_h^{(i)} - \Delta_x u_h^{(i)}\]

for the truncation error, and \(\Delta_x\) for the adjoint of \(\Delta\) defined by

\[\Delta_x V(x) = \sum_{y \in \Omega_h} I_h(y, x)V(y).\]

Recall by (2.6) and our smoothness assumption on \(u^{(i)}\) that

\[|\tau_h u_h^{(i)}| \leq C_i h^4, \quad \text{on } \Omega', \quad \leq C_i h^2, \quad \text{on } \Omega^{(2)} \cup \Omega^{(3)}.\]

However, on \(\Omega^{(2)} \cup \Omega^{(3)}\) both \(U_h^{(i)}\) and \(u_h^{(i)}\) are bounded by \(C_i h\), while the number of points in \(\Omega^{(2)} \cup \Omega^{(3)}\) is only proportional to \(h^{-1}\). From these considerations, we see that the first three terms on the right side of (8.12) are bounded by \(C_i h^4\). As for the remaining term,

\[\Delta u_h^{(i)}(x) - \Delta_x u_h^{(i)}(x)\]

vanishes for \(x \in \Omega' \cup \Omega^{(2)} \cup \Omega^{(3)}\), and is bounded by

\[Ch^{-2} \max_{\Omega' \cup \Omega^{(2)} \cup \Omega^{(3)}} |u_h^{(i)}| \leq C_i h^{-1}\]

for \(x \in \Omega' \cup \Omega^{(2)} \cup \Omega^{(3)}\). Again noting that the number of points in \(\Omega' \cup \Omega^{(2)} \cup \Omega^{(3)}\) is only proportional to \(h^{-1}\), the last term on the right of (8.12) is bounded by

\[C_i \max_{\Omega' \cup \Omega^{(2)} \cup \Omega^{(3)}} |U_h^{(i)} - u_h^{(i)}|.
\]

Thus, using (8.11) we have the inequality

\[|\lambda_h^{(i)} - \lambda^{(i)}| \leq C_i \left[ \max_{\Omega' \cup \Omega^{(2)} \cup \Omega^{(3)}} |U_h^{(i)} - u_h^{(i)}| + h^4 \right].\]
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\[ C, |\lambda | - \lambda |, \text{or if } |x - \partial \Omega| \leq Ch, (5.17) \text{shows the last term bounded by } C, h |\lambda | - \lambda |. \text{Using (8.3), (5.16) and Schwarz's inequality bound the middle term on the right by } ||U_h^{(i)} - u_h^{(i)}||_h, \text{or, if } |x - \partial \Omega| \leq Ch, (5.17) \text{bounds it by } C, h \max_{\Omega_h} |U_h^{(i)} - u_h^{(i)}|. \text{In summary,}

\begin{align*}
\text{(8.15)} & \quad \max_{\Omega_h} |U_h^{(i)} - u_h^{(i)}| \leq C, (||U_h^{(i)} - u_h^{(i)}||_h + |\lambda | - \lambda | + h^4), \\
\text{(8.16)} & \quad \max_{\Omega_h \cap \Omega_h^{(i)} \cup \Omega_h^{(i)}} |U_h^{(i)} - u_h^{(i)}| \leq C, \left( h \max_{\Omega_h} |U_h^{(i)} - u_h^{(i)}| + h |\lambda | - \lambda | + h^4 \right).
\end{align*}

Finally, we use Parseval's identity and (8.9) to conclude that

\[ ||U_h^{(i)} - u_h^{(i)}||_h = h^2 \sum_{\Omega_h \cap \Omega_h^{(i)}} |U_h^{(i)} - u_h^{(i)}|^2 + \sum_{j \neq i, \ldots, i + m} |\langle U_h^{(i)} - u_h^{(i)}, V_h^{(j)} \rangle_h|^2, \]

and by a straightforward computation

\[ \langle u_h^{(i)} - \lambda |, U_h^{(i)} - u_h^{(i)}, V_h^{(j)} \rangle_h = \langle \lambda u_h^{(i)} - \lambda |, U_h^{(i)} - u_h^{(i)}, \overline{H_h^{(i)}} \rangle_h, \]

where \( \overline{H_h^{(i)}} \) is defined by

\[ \Delta_h \overline{H_h^{(i)}}(x) = 0, \quad x \in \Omega_h, \quad \overline{H_h^{(i)}}(x) = U_h^{(i)}(x) - u_h^{(i)}(x), \quad x \in \Omega_h^{(i)} \cup \Omega_h^{(j)}. \]

It follows that

\[ ||U_h^{(i)} - u_h^{(i)}||_h \leq C, \left( \max_{\Omega_h \cap \Omega_h^{(i)} \cup \Omega_h^{(i)}} |U_h^{(i)} - u_h^{(i)}| + |\lambda | - \lambda | + h^4 \right). \]

Combining (8.13), (8.15), (8.16), and (8.17) yields the proof of Theorem 8.1.

Let us observe some simple consequences of Theorem 8.1. Since the \( \lambda | \) are real, we have

\[ |\text{Re} \lambda | - \lambda | | \leq Ch^4. \]

Also, when \( \lambda | \) is simple, \( \lambda | \) will be real for \( h \) sufficiently small. This is because the matrix \( [l_h(x, y)]_{x, y} \) is real. Thus, if \( \lambda | \) were complex, its conjugate \( [\lambda \overline{|}] \) would also be a distinct eigenvalue of \( \Delta_h \) converging to \( \lambda | \). But this is impossible, since \( [\lambda | \overline{]} \) must converge to some \( \lambda | \neq \lambda | \).

We normalized \( U_h^{(i)} \) by requiring \( ||U_h^{(i)}||_h = 1 \). This determines \( U_h^{(i)} \) only up to a multiplicative constant of modulus 1. If we specify this constant by requiring that \( \langle U_h^{(i)}, V_h^{(j)} \rangle_h \geq 0 \), then when \( \lambda | \) is simple, \( u_h^{(i)} \) is a real multiple of \( u^{(i)} \), as can be seen from (8.9).

Theorem 8.1 shows that \( U_h^{(i)} \) approximates to \( O(h^4) \) an eigenfunction \( u_h^{(i)} \) which depends on \( h \). Properly normalized, however, \( U_h^{(i)} \) will approximate to \( O(h^4) \) an eigenfunction \( u^{(i)} \) such that \( \int \sigma | u^{(i)} |^2 dx = 1 \), independently of \( h \). In particular, when \( \lambda | \) is simple, \( U_h^{(i)} \) will approximate the unique normalized eigenfunction \( u^{(i)} \). This normalization is

\[ h^2 \sum_{x \in \Delta_h} \alpha_h(y) |U_h^{(i)}(y)|^2 = 1, \]

where \( \alpha_h \) is given in the appendix of [6]. For a proof, see [6, Corollary 6.2].

9. Forced Vibration Problems. Let us remark that all of the results of the previous sections hold for the problem

\[ \Delta u(x) + (q(x) + \lambda)u(x) = 0, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial \Omega, \]
where \( q \) is nonpositive and smooth on \( \Omega \), and for the discrete Green's function \( G_h \) defined by

\[
(\Delta_h z + q(x))G_h(x, y) = -h^{-2} \delta(x, y), \quad x, y \in \Omega_h.
\]

The proofs require only that the additional term \( q \) be carried along throughout. We make this remark because we next wish to consider the problem

\[
(9.3) \quad \Delta u(x) + r(x)u(x) = F(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega,
\]

for \( F \) and \( r \) given smooth functions on \( \Omega \). Problem (9.3) is a forced vibration problem and an \( O(h^2) \) analogue of it was studied by Bramble in [1].

Let us rewrite (9.3) in the form

\[
(9.4) \quad \Delta u(x) + q(x)u(x) + \left( \sup_{y} r(y) \right) u(x) = F(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega,
\]

where \( q(x) = r(x) - \sup_{y} r(y) \leq 0 \) on \( \Omega \). A unique solution \( u \) of (9.3) or (9.4) exists if and only if \( \sup_{y} r \) is not an eigenvalue of the operator \( \Delta + q \). Now, we consider the difference approximation

\[
(9.5) \quad \Delta_h u_h(x) + r(x)u_h(x) = F(x), \quad x \in \Omega_h,
\]

where \( \Delta_h \) is the difference operator defined in Section 2. We prove:

**Theorem 9.1.** If (9.3) has a unique solution \( u \in C^6(\Omega) \), there are constants \( C, h_0 \) such that for \( h < h_0 \), (9.5) has a unique solution \( u_h \) for which

\[
\max_{\Omega_h} |u_h - u| < Ch^4.
\]

**Proof.** Let \( G_h \) be the discrete Green's function defined in (9.2). Then, for \( x \in \Omega_h \),

\[
|u_h(x) - u(x)| = \left| h^2 \sum_{y \in \Omega_h} G_h(x, y)[\Delta_h u(y) + q(y)u(y) - \Delta_h u_h(y) - q(y)u_h(y)] \right|
\leq \sup_{y \in \Omega_h} |q(y)| h^2 \sum_{y \in \Omega_h} |G_h(x, y)| |u_h(y) - u(y)| + h^2 \sum_{y \in \Omega_h} |G_h(x, y)| |r(y)u(y)|.
\]

Therefore, using (5.13) and (5.14) for \( G_h \) of (9.2) and (2.5),

\[
|U_h(x) - u(x)| \leq C \left[ h^2 \sum_{y \in \Omega_h} |G_h(x, y)| |U_h(y) - u(y)| + h^4 \right].
\]

Employing (5.17), this yields

\[
(9.7) \quad \max_{U_h \in \mathbb{R}^{n \times n}} |u_h - u| \leq C \left[ h \max_{U_h} |U_h - u| + h^4 \right],
\]

while (5.16) and Schwarz's inequality yield

\[
(9.8) \quad \max_{U_h} |U_h - u| \leq C \left[ |U_h - u| \right] h^4.
\]

From (9.7) and (9.8), we see

\[
\max_{U_h \in \mathbb{R}^{n \times n}} |U_h - u| \leq C \left[ \max_{U_h \in \mathbb{R}^{n \times n}} |U_h - u| \right] h^4.
\]
which implies

\begin{equation}
\| U_h - u \|_h \leq C[\| U_h - u \|_{h} + h^4].
\end{equation}

Finally, we complete the proof by using Parseval's identity to estimate

\begin{equation}
\| U_h - u \|_{h} = \left[ \sum_i |\langle U_h - u, V_h^{(i)} \rangle_h|^2 \right]^{1/2},
\end{equation}

where \( V_h^{(i)} \) is the eigenvector associated with \( \mu_h^{(i)} \) in the symmetric problem

\[ \Delta_h V_h^{(i)}(x) + (q(x) + \mu_h^{(i)}) V_h^{(i)}(x) = 0, \quad x \in \Omega', \quad V_h^{(i)}(x) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}. \]

Define \( H_h \) by

\[ \Delta_h H_h(x) + q(x) H_h(x) = 0, \quad x \in \Omega', \quad H_h(x) = U_h(x) - u(x), \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}. \]

From (5.21), we have

\[ \max_{\Omega_h} |H_h| \leq C \max_{\Omega_h \in \Omega_h^{(1)} \cup \Omega_h^{(2)}} |U_h - u|, \]

or, employing (9.7), (9.8), (9.9),

\begin{equation}
\max_{\Omega_h} |H_h| \leq C|h| \| U_h - u \|_{h} + h^4.
\end{equation}

Then, we have

\[ \mu_h^{(i)} \langle U_h - u, V_h^{(i)} \rangle_h = \langle H_h + u - U_h, (\Delta_h + q)V_h^{(i)} \rangle_h + \mu_h^{(i)} \langle H_h, V_h^{(i)} \rangle_h \]

\[ = \langle (\Delta_h + q)(H_h + u - U_h), V_h^{(i)} \rangle_h + \mu_h^{(i)} \langle H_h, V_h^{(i)} \rangle_h \]

\[ = (\sup r)(U_h - u, V_h^{(i)} \rangle_h - \langle r, u, V_h^{(i)} \rangle_h + \mu_h^{(i)} \langle H_h, V_h^{(i)} \rangle_h. \]

Now, since \( \sup r \) is not an eigenvalue \( \lambda^{(i)} \) of \( \Delta + q \) and \( \mu_h^{(i)} \to \lambda^{(i)} \) as \( h \to 0 \), there are constants \( C, h_0 \) such that for \( h < h_0 \),

\[ \max_i |\mu_h^{(i)} - \sup r|^{-1} < C, \quad \max_i \mu_h^{(i)} / |\mu_h^{(i)} - \sup r| < C, \]

and so

\[ |\langle U_h - u, V_h^{(i)} \rangle_h| \leq C[|\langle r, u, V_h^{(i)} \rangle_h| + |\langle H_h, V_h^{(i)} \rangle_h|]. \]

Using this in (9.10), we see that

\[ \| U_h - u \|_{h} \leq C[\| r, u \|_{h} + |H_h|_{h}] \leq C[h^4 + h \| U_h - u \|_{h}], \]

by (9.11), from which it follows that

\[ \| U_h - u \|_{h} \leq C h^4, \]

completing the proof.

Let us remark that by employing the results of [6], the above technique of proof will show that a unique solution of the forced vibration problem:

\begin{equation}
\sum_{i=1}^{r} \frac{\partial}{\partial \lambda_i} \left( u_i(\xi) \frac{\partial u_i(\xi)}{\partial \lambda_i} \right) + r(\xi) u(\xi) = f(\xi), \quad \xi \in \Omega,
\end{equation}

\[ u(\xi) = 0, \quad \xi \in \partial \Omega, \]

\[ u^{(i)}(\xi) = 0, \quad \xi \in \partial \Omega, \]
can be approximated to $O(h^2)$ by using the symmetric difference scheme given in [6] at the beginning of Section 7.

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