MATHEMATICS OF COMPUTATION, VOLUME 25, NUMBER 115, JULY, 1971

# Exponential Chebyshev Approximation on Finite Subsets of [0, 1]

By Bernard H. Rosman

**Abstract.** In this note the convergence of best exponential Chebyshev approximation on finite subsets of [0, 1] to a best approximation on the interval is proved when the function to be approximated is continuous and when the union of the finite subsets is dense in [0, 1].

**1. Introduction.** In this note we study the convergence of best exponential Chebyshev approximation on finite subsets of [0, 1] to a best approximation on the interval. This problem has been considered for linear approximation [1] and, recently, for generalized rational approximation in [2].

Let $X_r$ be a set of $r$ distinct points in [0, 1], containing the endpoints (a common computational situation). We assume that the sequence of subsets $\{X_r\}$ fills up the interval in the sense that, given $x \in [0, 1]$, there is an $x_r \in X_r$ such that $\{x_r\} \to x$.

Following Rice [3, Chapter 8], we approximate $f \in C([0, 1])$ by exponential functions of the form

$$E(A, x) = \sum_{i=1}^{k} \left( \sum_{j=0}^{m_i} p_{ij} x^j \right) e^{t_i x}, \tag{1}$$

where $|p_{ij}| < \infty$, $|t_i| < \infty$ and $\sum_{i=1}^{k} (m_i + 1) \leqq n$, $n$ a fixed positive integer.

For each set $X_r$, we define the usual seminorm on $C([0, 1])$, corresponding to $X_r$, by

$$||f||_{X_r} = \sup_{x \in X_r} |f(x)|$$

for all $f$ in $C([0, 1])$. We denote by $E(A_r, x)$ and $E_r(x)$ the best approximation to $f(x)$ on $X_r$, i.e. the best approximation to $f$ with respect to the seminorm corresponding to $X_r$. Norm signs without subscripts denote the usual Chebyshev norm.

It is known [3] that best approximation need not exist on finite point sets. However, we assume existence, a reasonable assumption in many computational situations. Moreover, Rice solves a special case of this problem through the use of pseudo functions [3, pp. 65–69]. An extension of this technique to handle general exponential approximation is under investigation.

**2. The Convergence Theorem.** The main result of this note is

THEOREM 1. *Let $E^*$ be a best exponential approximation to $f$ on* [0, 1]. *Then*

$$||f - E_r||_{X_r} \to ||f - E^*||.$$

Received January 22, 1970, revised September 9, 1970.

*AMS* 1970 *subject classifications.* Primary 41A30, 41A50; Secondary 65D10.

*Key words and phrases.* Chebyshev approximation, exponential functions, varisolvence, pointwise convergence, uniform convergence.

*Proof.* First, suppose that $\{||E_r||\}$ is a bounded sequence. Then by a theorem in Rice [3, Chapter 8], $\{E_r(x)\}$, or a subsequence thereof, converges pointwise to a possibly discontinuous function

$$\begin{aligned} E(x) &= E_0(x), & 0 < x < 1, \\ &= e_0, & x = 0, \\ &= e_1, & x = 1, \end{aligned}$$

where $E_0(x)$ is an exponential of form (1). We claim that $E_0$ is a best approximation to $f$ on $[0, 1]$. For if not, there exists a point $x$ such that $|f(x) - E_0(x)| > ||f - E^*||$. By continuity, we may assume that $x \in [\delta, 1 - \delta]$ where $\delta > 0$ is sufficiently small. Let $\{x_r\} \to x$ with $x_r \in X_r$, $\epsilon > 0$, and let $y_1, \cdots, y_n$ be in $[\delta, 1 - \delta]$. By the definition of varisolvence, there is a $\delta(\epsilon)$ such that the inequality $|Y_i - E_0(y_i)| < \delta(\epsilon)$ implies that there is a function $E(A, y)$ satisfying $E(A, y_i) = Y_i$ and $|E(A, y) - E_0(y)| < \epsilon$. But since $E_r \to E_0$ pointwise on $[\delta, 1 - \delta]$, for $r$ large enough, $|E_r(y_i) - E_0(y_i)| < \delta(\epsilon)$. Hence, there are functions $\{E(A_r, y)\}$ such that $E(A_r, y_i) = E_r(y_i)$. But since these interpolating functions are unique ($m$ is the degree of solvence), $E(A_r, y) = E_r(y)$ for all $y$. Therefore, $|E_r(y) - E_0(y)| < \epsilon$ for $y$ in $[\delta, 1 - \delta]$. Therefore, by a standard inequality, $|f(x_r) - E_r(x_r)| \to |f(x) - E_0(x)|$. Hence, for sufficiently large $r$, $|f(x_r) - E_r(x_r)| > ||f - E^*||$. This contradicts the fact that $E_r$ is a best approximation on $X_r$ and hence $E_0$ is a best approximation.

Suppose now that $\{||E_r||\}$ is an unbounded sequence. Following Dunham [2], define $B_r(x) = E_r(x)/||E_r||$. Then $\{||B_r||\}$ is a bounded sequence and, by the aforementioned result of Rice, we may assume that $\{B_r(x)\}$ converges pointwise to a function of the form

$$\begin{aligned} B(x) &= B_0(x), & 0 < x < 1, \\ &= b_0, & x = 0, \\ &= b_1, & x = 1, \end{aligned}$$

where $B_0(x)$ is an exponential of form (1). Now, assume that $b_0 = B_0(0)$ and $b_1 = B_0(1)$. The $B(x)$ is an exponential and, using varisolvence as before, it follows that $\{B_r\}$ converges uniformly to $B$. Since $||B_r|| = 1$, there exists $y \in [0, 1]$ and a neighborhood $N$ of $y$ such that $m = \inf_{x \in N} B(x) > 0$ and $B_r(x) > m/2$ for $r$ sufficiently large. Hence, $\inf_{x \in N} E_r(x) \to \infty$ as $r \to \infty$ and, for large enough $r$, there exists $x_r \in X_r$ such that $E_r(x_r) > 2||f||$. This contradicts $E_r$ being a best approximation to $f$ on $X_r$ since then

$$||f - E_r||_{X_r} > ||f|| \geqq ||f||_{X_r} = ||f - 0||_{X_r}.$$

It remains to consider the case where $B(x)$ has an endpoint discontinuity. Assume without loss of generality that $B(x) = B_0(x)$, $0 < x \leqq 1$, and $b_0 > B_0(0)$. If $B_0 \not\equiv 0$, there exists $y \in [\delta, 1 - \delta]$, $\delta > 0$, such that $B_0(y) \neq 0$ and $\{B_r(y)\} \to B_0(y)$. Using the varisolvence of $B_0$ on this interval, we again contradict $E_r$ being best on $X_r$. If $B_0 \equiv 0$, we reach the same contradiction on $E_r$ since then $\{E_r(0)\}$ is unbounded. This concludes the proof.

If we cannot assume that the endpoints are in $X_r$, the proof goes through without change except for the last case. If $B_0 \equiv 0$ and if $0 \in X_r$ for only a finite set of $r$ values with $b > 0$, it follows from the properties of exponentials and the boundedness of $||E_r||_{X_r}$ that $\{E_r(x)\}$ is bounded on $[\delta, 1 - \delta]$, $\delta > 0$. Hence, $\{E_r(x)\}$ or a subsequence thereof converges pointwise to $E_0(x)$, an exponential, on (0, 1). From this reasoning, as before using varisolvence, $E_0$ is a best approximation to $f$ on [0, 1].

From the proof of Theorem 1 we have

COROLLARY. *The sequence* $\{E_r(x)\}$ *has a subsequence which converges pointwise except possibly at the endpoints to a best exponential approximation,* $E_0(x)$, *to* $f$ *on* [0,1]. *If the subsequence converges to* $E_0(0)$ *and* $E_0(1)$ *also, then the convergence is uniform.*

Department of Mathematics
Boston University
Boston, Massachusetts 02215

1. T. J. RIVLIN & E. W. CHENEY, "A comparison of uniform approximations on an interval and a finite subset thereof," *SIAM J. Numer. Anal.*, v. 3, 1966, pp. 311–320. MR **34** #4773.

2. C. B. DUNHAM, "Rational Chebyshev approximation on subsets," *J. Approximation Theory*, v. 1, 1968, pp. 484–487. MR **38** #6279.

3. J. R. RICE, *The Approximation of Functions.* Vol. 2: *Nonlinear and Multivariate Theory*, Addison-Wesley, Reading, Mass., 1969. MR **39** #5989.