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# A Note on Chowla's Function*

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**Abstract.** Iterates of a number-theoretic function, defined by $L(n) = \sigma(n) - (1 + n)$, are investigated empirically, for $n \leqq 10^5$. This search has yielded 9 reduced amicable pairs.

**1. Introduction.** Professor Chowla defined a number-theoretic function, $L(n)$, for $n > 1$.

$$(1) \qquad L(n) = \sigma(n) - (1 + n)$$

where $\sigma(n) = \sum_{d|n} d$. That is, $L(n)$ denotes the sum of the divisors of $n$ except $n$ and unity.

For $n$ prime, $L(n) = 0$. The $r$th iterate of $L(n)$ is denoted by

$$(2) \qquad L_r(n) = L(L_{r-1}(n)); \qquad L_1(n) = L(n).$$

Professor Chowla conjectured that the sequence of iterates defined above takes only a finite number of different values, and Nasir [1] verified the conjecture for $n \leqq 100$. Furthermore, he found that the sequence converges to zero except for $n = 48$, 75 and 92.

Lehmer, in his review [2] of Nasir's paper, defined the pair (48, 75) as amicable, because $L(48) = 75$ and $L(75) = 48$. In order to avoid confusion, we shall call these pairs as reduced amicable pairs and the pairs defined by the function $S(n) = \sigma(n) - n$ as amicable pairs. In this brief note, we intend to investigate empirically certain properties of $L(n)$ and to provide some evidence for the question whether the number of such reduced amicable pairs is finite.

**Results.** For $n \leqq 10^5$, it was found that there are only 9 reduced amicable pairs. These pairs are given in Table 1.** If $A(n)$ is the number of reduced amicable pairs of which the smaller number is less than $n$, then the distribution of $A(n)$ is as follows: $A(n) = 1$ for $n \leqq 10^2$, $A(n) = 2$ for $n \leqq 10^3$, $A(n) = 8$ for $n \leqq 10^4$ and $A(n) = 9$ for $n \leqq 10^5$. It is of interest to note that the number of amicable pairs for $n \leqq 10^5$ is 13, which is comparable to that found for the reduced amicable pairs.

Regarding the function $L_r(n) = 0$, it was found that in most cases $L_r(n) = 0$ and there are only 2151 values for which the iterates converge to a member of a reduced amicable pair. The frequency with which a given reduced pair is reached while

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** See editorial note at end of paper.

TABLE 1
*Frequencies of Various Reduced Amicable Pairs $n \leqq 10^5$*

| Amicable Pairs | | Frequency |
|---|---|---|
| (48, | 75) | 1138 |
| (140, | 195) | 430 |
| (1050, | 1925) | 42 |
| (1575, | 1648) | 226 |
| (2024, | 2295) | 156 |
| (5775, | 6128) | 70 |
| (8892, | 16587) | 63 |
| (9504, | 20735) | 22 |
| (62744, | 75495) | 4 |

TABLE 2
*Lowest Values of n, for a Given r such that $L_r(n) = 0$*

| $r$ | $n$ | $r$ | $n$ |
|---|---|---|---|
| 1 | 2 | 19 | 3344 |
| 2 | 4 | 20 | 3888 |
| 3 | 8 | 21 | 5360 |
| 4 | 15 | 22 | 8895 |
| 5 | 12 | 23 | 11852 |
| 6 | 27 | 24 | 25971 |
| 7 | 24 | 25 | 23360 |
| 8 | 36 | 26 | 38895 |
| 9 | 90 | 27 | 35540 |
| 10 | 96 | 28 | 35595 |
| 11 | 245 | 29 | 36032 |
| 12 | 288 | 30 | 53823 |
| 13 | 368 | 31 | 47840 |
| 14 | 676 | 32 | 62055 |
| 15 | 1088 | 33 | 59360 |
| 16 | 2300 | 34 | 83391 |
| 17 | 1596 | 35 | 70784 |
| 18 | 1458 | | |

iterating $L_r(n) = 0$ is given in Table 1. Out of the total number of 2151 values where $L_r(n) \neq 0$, for some finite $r$, the smallest pair (48, 75) is reached 1140 times. The frequency with which these pairs appear decreases rapidly. The frequency for the pair (1050, 1925) is unexpectedly low.

It would be of interest to find reduced amicable triplets or groups of higher order. A reduced amicable triplet is defined to be a set of three distinct positive integers $n, m, p$, such that $L(n) = m$, $L(m) = p$ and $L(p) = n$. Similarly, groups of higher order are defined. For $n \leqq 10^5$, there are no triplets or groups of higher order.

**Bounds on the Number of Iterations.** For a given $r$, the smallest values of $n$, such that $L_r(n) = 0$, were recorded. A table of such values of $n$ suggest, for any $n$,

$$1 \leqq r \leqq c \ln(n); \qquad c = 3.2 \quad \text{for } n \leqq 10^5.$$

For the purpose of detailed comparison of the function $L(n) = \sigma(n) - (1 + n)$ with $S(n) = \sigma(n) - n$, it would be of interest to compare the upper bound for the iterations of $S(n)$ such that $S_r(n) = 1$. It is known that $S_r(n)$ is bounded for $2 \leqq n \leqq 275$. The verification for higher $n$ is very tedious. For $n = 276$, $S_r(n) = 1$, for $r > 119$ [3] and $S_{189}(936) = 1$. This suggests that the corresponding value of $c$ for the iterations of $S_r(n)$ is considerably higher.

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1. A. R. Nasir, "On a certain arithmetic function," *Bull. Calcutta Math. Soc.*, v. 38, 1946, p. 140. MR **8**, 445.
2. D. H. Lehmer (Reviewer), *Math. Rev.*, v. 8, 1948, p. 445.
3. H. Cohen, "On amicable and sociable numbers," *Math. Comp.*, v. 24, 1970, pp. 423–429.

Editorial note. While this paper was in press, we learned of a short note by Mariano Garcia, "Números Casi Amigos y Casi Sociables," that appeared in Revista Annal, año 1, October 1968, *Asociacion Puertorriqueña de Maestros de Matematicas*. Garcia's table on page 7 therein gives the same nine pairs listed in Table 1, and no others. The frequencies in Table 1 and the data in Table 2 are not given.