Minimal Quadratures for Functions of Low-Order Continuity

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Abstract. An analog of Wilf's quadrature is developed for functions of low-order continuity. This analog is used to demonstrate that the order of convergence of Wilf's quadrature is at least 1/n.

1. Introduction. From the work done in minimal norm quadratures for Hilbert spaces of analytic functions by Wilf [7], Barnhill [1], Eckhardt [2], Richter [6], and others, it is natural to consider an extension of this concept for functions of low-order continuity. In this paper, we consider functions with a uniformly convergent Fourier-Chebyshev expansion on the interval \([-1, 1]\)

\[ f(x) = \sum_{i=0}^{\infty} a_i T_i(x), \]

where \(T_i(x)\) is the \(i\)th degree Chebyshev polynomial of the first kind and the prime on the sum indicates the first term is to be halved. We also restrict \(f(x)\) to have the property that \(\sum_{i=0}^{\infty} |a_i|\) converges, e.g. when \(f'(x)\) is of bounded variation on \([-1, 1]\).

For error bounds of Gaussian quadrature for functions of this type, see Rabinowitz [5].

2. Minimal Quadratures. Let \(\sum_{i=0}^{n} H_i f(x_i)\) be an \((n + 1)\)-point quadrature formula. We define \(R_n(f) = \int_{-1}^{1} f(x) \, dx - \sum_{i=0}^{n} H_i f(x_i)\) and note from the expansion of \(f(x)\) that \(R_n(f) = \sum_{i=0}^{n} a_i R_n(T_i)\). Using both the triangle and Schwarz inequalities we obtain the error estimate

\[ |R_n(f)| \leq \left( \sum_{i=0}^{k} a_i^2 \right)^{1/2} \left( \sum_{i=0}^{k} R_n(T_i)^2 \right)^{1/2} + \sum_{i=k}^{n} |a_i R_n(T_i)| \]

where the double prime indicates both first and last terms are to be halved.

If \(f(x)\) satisfies mild smoothness restrictions (cf. Elliott, [3]), then the coefficients \(a_i\) satisfy \(|a_i| \leq C/i^2\). In this case, since \(R_n(T_i)\) is bounded for \(i \geq k\), the last term of the inequality is of order \(1/k\). Thus, it appears worthwhile to consider, as in Wilf [7], minimizing \(W(n, k)\) where

\[ W(n, k) = \sum_{i=0}^{k} R_n(T_i)^2. \]

Received October 9, 1970, revised March 8, 1971.

AMS 1970 subject classifications. Primary 65D30.

Key words and phrases. Wilf's quadrature, optimal quadrature, order of convergence, low-order continuity.

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We note \( W(n, k) = 0 \) for \( k < 2n + 2 \), since the problem is solved by Gauss-Legendre quadrature.

To minimize \( W(n, k) \), of course, we must solve the \( 2n + 2 \) simultaneous equations \( \partial W(n, k)/\partial H_s = 0 \) and \( \partial W(n, k)/\partial x_s = 0 \), \( 0 \leq s \leq n \). An analytic solution does not seem feasible, so we consider the less restrictive problem of choosing weights to minimize \( W(n, k) \) with a given fixed set of nodes. In doing so we are able to answer a question posed by Wilf (see Section 4).

Solving \( \partial W(n, k)/\partial H_s, 0 \leq s \leq n \), leads to the system

\[
\sum_{i=0}^{k} \alpha_i T_i(x_s) = 0; \quad s = 0, \ldots, n.
\]

Setting

\[
\alpha_i = \begin{cases} 
\int_{-1}^{1} T_i(x) \, dx = -2/(i^2 - 1), & i \text{ even}, \\
= 0, & i \text{ odd},
\end{cases}
\]

thus (3) becomes

\[
\sum_{i=0}^{k} \alpha_i T_i(x_s) = \sum_{i=0}^{n} H_i T_i(x_s) T_i(x_s).
\]

If \( H^*_0, \ldots, H^*_n \) satisfy (4), then \( \sum_{i=0}^{n} H^*_i f(x_i) \) is called a minimal quadrature.

Let \( g_a(x) = \sum_{i=0}^{k} \alpha_i T_i(x) \) and \( f_a(x) = \sum_{i=0}^{n} R_a(T_i) T_i(x) \). Then,

\[
R_a(g_a) = \sum_{i=0}^{k} \alpha_i R_a(T_i) = \int_{-1}^{1} f_a(x) \, dx = R_a(f_a) - \sum_{i=0}^{n} H_i f_a(x_i).
\]

If \( H_0, \ldots, H_n \) is a solution of (4), then (3) the quadrature sum is zero. Further, as \( R_a(f_a) = W(n, k) \), we have \( R_a(g_a) = W(n, k) \) so \( W(n, k) \), for any minimal quadrature, is the error made in approximating the integral of \( g_a(x) \). We note here that \( \sum_{i=0}^{n} \alpha_i T_i(x) \) is the Fourier-Chebyshev expansion for \( F(x) = \sqrt{1-x^2} \) on \([-1, 1] \), and since \( F(x) \) is continuous and of bounded variation the series is uniformly convergent.

Let \( H \) denote the \((n+1)\)-dimensional vector \( H = (H_0, \ldots, H_n) \) and define \( \varphi : E^{n+1} \to E^{n+1} \) by \( \varphi(H) = (R_a(T_0), \ldots, R_a(T_n)) \), where \( R_a(T_i) = \alpha_i - \sum_{i=0}^{n} H_i T_i(x_s) \).

It is immediate from Hilbert space properties that there is a unique point \( H^* \) in \( E^{n+1} \) such that \( ||\varphi(H^*)||_2 \) is minimal. Thus, the existence of a unique minimal quadrature is guaranteed.

3. Special Case. When \( k = n \), the minimal quadrature is of course the interpolatory quadrature on \( x_0, \ldots, x_n \). In the case \( x_i = \cos(i\pi/n) \), the interpolatory quadrature is Clenshaw-Curtis quadrature. If we use the well-known orthogonality properties for \( T_i(x) \) in (4) with \( x_i = \cos(i\pi/n) \), we obtain immediately \( \frac{1}{2n} H_i = g_a(x_i) \), \( i = 1, \ldots, n-1 \), and \( nH_0 = g_a(x_0) \) for \( i = 0 \) or \( n \). These are the same expressions found by Imhof [4], which he used to show the Clenshaw-Curtis weights were positive.

4. Improvement of a Result of Wilf. In [7] Wilf minimizes \( W_n = \sum_{i=0}^{n} R_a(x_i)^2 \). Let \( R^*_a \) denote the remainder for optimal quadrature in the set of functions analytic in \( |z| < 1 \) and \( \mathbb{Z} \) on the unit circle, and let
Thus $|R_{n}(f)| \leq W_{n}^{1/2} ||f||$. Wilf was unable to give explicit solutions for the weights and nodes, but was able to show that $W_{n}$ is the magnitude of the error in integrating $x^{-1} \log(1 - x)^{-1}$ by the minimal formula. He derives the result $W_{n} \leq O(\ln(n)/n)$ and leaves as an open question whether this result can be improved. On $[0, 1]$ the Clenshaw-Curtis weights and nodes are, respectively,

$$w_{i} = \frac{4}{n + 1} g_{n+1}(x_{i}), \quad i = 1, \ldots, n,$$

$$= \frac{2}{n + 1} g_{n+1}(x_{i}), \quad i = 0 \text{ or } n + 1;$$

For ease of computation, and since $g_{n+1}(x)$ is uniformly convergent to $\frac{1}{2} \pi (1 - x^{2})^{1/2}$ on $[0, 1]$, we shall use instead the weights

$$H_{i} = \frac{2\pi}{n + 1} (1 - (x_{i})^{2})^{1/2}, \quad i = 1, \ldots, n,$$

$$= \frac{\pi}{n + 1} (1 - (x_{i})^{2})^{1/2}, \quad i = 0 \text{ or } n + 1,$$

and we note $H_{0} = x_{n+1} = 0$.

Since Clenshaw-Curtis quadrature is exact for polynomials of degree less than $n + 2$,

$$\sum_{s=0}^{n+1} w_{s}(x_{s})^{2} = \frac{1}{k + 1} \quad \text{for } 0 \leq k \leq n + 1.$$
Now, for $k > n$, define

$$Q_n(x^k) = \sum_{i=1}^{n} H_i x^k \leq \frac{2\sqrt{2} \pi}{n + 1} \sum_{i=1}^{n} \sin \left( \frac{s\pi}{2n + 2} \right) \left( \cos \left( \frac{s\pi}{2n + 2} \right) \right)^{2k}. \tag{6}$$

In what follows we will show that $Q_n(x^k) = O(1/k)$ for $k > n$. We first show that if $n$ is sufficiently large and $k \geq (n + 1)^2$, then $kQ_n(x^k)$ is bounded independently of $k$. We then show by an integral bound that $Q_n(x^k) = O(1/k)$ for the remaining $k$ in $(n(n + 1)^2)$.

First let us assume that $k \geq (n + 1)^2$, then

$$y(k) = kQ_n(x^k) \leq \sum_{i=1}^{n} \sin \left( \frac{s\pi}{2n + 2} \right) V_i(k), \tag{7}$$

where $V_i(k) = k(\cos(s\pi/(2n + 2)))^{2k}$. Then it is easily verified that for $n$ sufficiently large (say $n \geq M$) and $k \geq (n + 1)^2$, $V_i(k) < 0$, and thus $y(k)$ is bounded for all such $k$.

Now we consider a fixed $n \geq M$ and any $k < (n + 1)^2$. We define

$$z(s) = \left( \frac{2\sqrt{2}\pi}{n + 1} \right) \sin \left( \frac{s\pi}{2n + 2} \right) \left( \cos \left( \frac{s\pi}{2n + 2} \right) \right)^{2k}. \tag{8}$$

Then $z'(s^*) = 0$ for $\tan(s^*\pi/(2n + 2)) = (1/2k)^{1/2}$, and $s^*$ is unique in $(0, n + 1)$. Thus if $m$ is the greatest integer in $s^*$, and since $z(s) \geq 0$ in $(0, n + 1)$ and is maximal at $s^*$

$$\sum_{s=m+1}^{n} z(s) \leq \int_{0}^{n+1} z(t) \, dt < \frac{2\sqrt{2}}{k}. \tag{9}$$

Since $z(s^*) \leq (2\sqrt{2}\pi/(n + 1))(2k + 1)^{-1/2}$, then

$$Q_n(x^k) \leq \sum_{i=1}^{n} z(s) \leq \frac{2\sqrt{2}}{k} + 2z(s^*) < \frac{2(\sqrt{2} + 2\pi)}{k}. \tag{10}$$

Thus, combining these two cases with $C_2 = 2(\sqrt{2} + 2\pi)$,

$$\sum_{k=n+1}^{\infty} R_n(x^k)^2 \leq \sum_{k=n+1}^{\infty} (1/(k + 1)^2 + 2C_2/k(k + 1) + C_2^2/k^3) \tag{11}$$

$$= O(1/n).$$

Combining this with (5) we get the desired result,

$$W_n \leq \sum_{k=0}^{\infty} R_n(x^k)^2 = O(1/n). \tag{12}$$

Reflection on the magnitude of $R_n^*(x^k)^2$, i.e. $(1/(k + 1) - Q_n^*(x^k))^2$, and the number of free parameters available leads us to conjecture that $O(1/n)$ is the best possible bound for $W_n$.

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