

# Infinite Sums of Roots for a Class of Transcendental Equations and Bessel Functions of Order One-Half

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**Abstract.** The roots of Bessel functions of order one-half are special cases of roots of transcendental equations of the form  $\tan z = A(z)/B(z)$ , where  $A(z), B(z)$  are polynomials and  $A(z)/B(z)$  is odd. We prove that the function  $f(z) = B(z) \sin z - A(z) \cos z$ ,  $f(z)$  even or odd, satisfies the conditions of Hadamard's factorization theorem, and derive recurrences for sums of the form  $S_l = \sum_{k=1}^{\infty} \alpha_k^{-2l}$ ,  $l = 1, 2, \dots$ , where the  $\alpha_k$ 's are the nonzero roots of  $f(z)$ . We also prove under what conditions on  $A(z)$  and  $B(z)$  is  $S_l = \pi^{-2l-2} \zeta(2l+2)$  or  $S_l = \pi^{-2l-2} \zeta(2l+2)(2^{2l+2} - 1)$ , where  $\zeta$  is the Riemann zeta function. We prove that, although Bessel functions of positive half-order,  $J_{l+1/2}$ , have only real roots, perturbation of any one of its coefficients introduces nonreal roots for  $l > 2$ .

1. Introduction. We are interested in sums of the form

$$(1.1) \quad S_k = \sum_{n=1}^{\infty} \alpha_n^{-2k-2},$$

where the  $\alpha_n$ 's are the nonzero roots of a function of the type

$$(1.2) \quad f(z) = B_m(z) \sin z - A_n(z) \cos z,$$

where  $B_m(z), A_n(z)$  are polynomials of order  $m, n$ , respectively,  $m \neq n$ , and  $f(z)$  is either even or odd. Since the roots occur in pairs,  $\pm\alpha$ , we take only one of each pair.

Two special cases of (1.2),  $B_m(z) = 1, A_n(z) = kz$ , and  $B_m(z) = z, A_n(z) = -k$ ,  $k$  a nonzero real constant, have been treated in [1] using Sturm-Liouville theory.

We shall show that (1.2) has a discrete sequence of roots,  $\alpha_n$ , with  $\alpha_n^2 \rightarrow \infty$ . As a special case, we get the Bessel functions  $J_{l+1/2}(z)$  and  $J_{-l-1/2}(z)$ , for  $l > 0$ . For  $J_{l+1/2}(z)$  we prove, by using  $S_0$ , that if  $l > 2$ , although  $J_{l+1/2}(z)$  itself has only real roots, perturbations of any one of its coefficients (written in the form (1.2)), introduce nonreal roots.

## 2. Main Theorem.

LEMMA 1. Let

$$(2.1) \quad f(z) = B_m(z) \sin z - A_n(z) \cos z, \quad m \neq n,$$

where  $B_m(z), A_n(z)$  are polynomials of order  $m, n$ , respectively, which have no common root.

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Let  $z = x + iy$ . Then

(1) There exists a  $Y > 0$ , such that if  $y > Y$ ,  $f(z)$  has no roots.

(2) In any strip  $|x| \leq L < \infty$ , there can only be a finite number of roots of  $f(z)$ .

*Proof.* Part (1).  $f(z) = 0$  iff

$$(2.2) \quad \tan z = A_n(z)/B_m(z),$$

(including both sides  $= \infty$ ). For  $|z| \gg 1$ ,

$$(2.3) \quad |A_n(z)/B_m(z)| \sim C|z|^{n-m} = C|z|^k,$$

where  $k = n - m$  and  $C > 0$ . If  $|z| \gg 1$ , we must therefore have

$$(2.4) \quad |\tan z| \approx C|z|^k,$$

if  $z$  is a root. But

$$\begin{aligned} \tan z = \tan(x + iy) &= \frac{\tan x + i \tanh y}{1 - i \tan x \tanh y} \\ &= \frac{\sin x \cos x (1 - \tanh^2 y) + i \tanh y}{\cos^2 x + \sin^2 x \tanh^2 y}, \end{aligned}$$

from which it follows that

$$(2.5) \quad |\tanh y| \leq |\tan z| \leq \frac{1}{2} + \frac{1}{2 \tanh^2 y},$$

i.e.,  $\tan z$  is bounded away from zero and infinity, if  $y \neq 0$ . If  $|y| \gg 1$ , then (2.5) implies that  $|\tan z| \approx 1$ . But if  $|y| \gg 1$ , then  $|z| \gg 1$ , and (2.4) holds for a root, i.e.,  $|\tan z| \gg 1$  for  $k > 0$ , and  $|\tan z| \ll 1$  for  $k < 0$ .

This concludes the proof of (1).

*Part (2).* Suppose we had an infinite number of roots for  $|x| \leq L < \infty$ . By Part (1), they would be in a bounded domain, and would have an accumulation point other than infinity. Since  $f(z)$  is an analytic entire function, it follows that  $f(z) \equiv 0$ , by Taylor's Theorem. This contradiction proves Part (2).

**Remark.** It follows from (2.4) that we do have an infinite sequence of roots, tending to  $\pm \infty$  on the real line, with the asymptotic values  $n\pi$  for  $k < 0$  and  $(n + \frac{1}{2})\pi$  for  $k > 0$ ,  $n$  an integer.

**LEMMA 2.** Let  $\lambda \equiv$  order of  $f(z)$ . Then  $\lambda \leq 1$ .

*Proof.* Let  $M(r) = \max_{|z|=r} |f(z)|$ . Then,  $M(r) \leq (n + m)C_1 r^{m+n} e^r$ , for  $r > 1$ , and  $C_1$  is the largest coefficient in absolute value of  $A_n(z)$  and  $B_m(z)$ .

$$\begin{aligned} \log M(r) &\leq \log[(n + m)C_1] + (m + n) \log r + r \\ &\leq C_2 r, \quad \text{for } r \text{ large enough and } C_2 > 0. \end{aligned}$$

So that,  $\log \log M(r) \leq \log C_2 + \log r$ , and therefore,

$$\lambda = \limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r} \leq 1 + \lim_{r \rightarrow \infty} \frac{\log C_2}{\log r} = 1.$$

**THEOREM 1.** Let  $f(z) = B_m(z) \sin z - A_n(z) \cos z$ , where  $B_m(z)$ ,  $A_n(z)$ , are polynomials of order  $m$ ,  $n$ , respectively, one being even and the other odd. Then

$$(2.6) \quad f(z) = C_0 z^q \prod_{k=1}^{\infty} (1 - z^2/\alpha_k^2),$$

where  $\alpha_k$  are the roots of  $f(z) = 0$ , such that  $|\alpha_1| \leq |\alpha_2| \leq \dots \leq |\alpha_n| \leq \dots$ ,  $q$  is the multiplicity of the root  $z = 0$ , and  $C_0 = f^{(q)}(0)/q!$ .

*Proof.* From Lemma 1 and the following remark we get that there exists an infinite sequence of roots of  $f(z) = 0$ , accumulating only at infinity. Also, the roots occur in pairs  $\pm\alpha$ . By Hadamard's factorization theorem [2, p. 22], we can write

$$(2.7) \quad f(z) = C_0 \exp\{g(z)\} z^q \prod_{k=1}^{\infty} (1 - z^2/\alpha_k^2), \quad C_0 \neq 0,$$

where  $q! C_0 = f^{(q)}(0)$ , and  $g(z)$  is a polynomial in  $z$  of order  $\leq \lambda$ , with  $g(0) = 0$ . By Lemma 2,  $\lambda \leq 1$ , and we get

$$(2.8) \quad g(z) = az.$$

Since  $f(z)$  is even or odd, we get that  $q$  must be even or odd with  $f(z)$ , and  $\exp\{g(z)\} = \exp\{g(-z)\}$ , i.e.,  $az = -az$  by (2.8), or  $a = 0$ , which proves (2.6).

3. Recurrences and Special Cases of  $S_k$ . Rewrite  $f(z)$  in Theorem 1 as

$$(3.1) \quad f(z) = b_{l+2m}(z) \sin z - a_{p+2n}(z) \cos z,$$

where

$$(3.2) \quad b_{l+2m}(z) = z^l \sum_{k=0}^m b_k z^{2k} = z^l B_m(z^2), \quad b_0 \neq 0, b_m \neq 0,$$

and

$$(3.3) \quad a_{p+2n}(z) = z^p \sum_{k=0}^n a_k z^{2k} = z^p A_n(z^2), \quad a_0 \neq 0, a_n \neq 0,$$

and  $l + p$  is an odd integer.

Since we are interested in the nonzero roots of  $f(z)$ , we can divide by  $b_0 z^{M \text{ in } (l,p)}$ . The two basic cases are therefore:

$$(3.4) \quad f(z) = B_m(z^2) \sin z - z^r A_n(z^2) \cos z, \quad r = 2t + 1 > 0,$$

and

$$(3.5) \quad f(z) = z^r B_m(z^2) \sin z - A_n(z^2) \cos z, \quad r = 2t + 1 > 0,$$

with  $b_0 = 1$  in both cases.

*Case 1.* Consider  $f(z)$  as in (3.4). Write

$$(3.6) \quad B_m(z^2) \sin z = \sum_{s=0}^{\infty} c_s z^{2s+1},$$

$$(3.7) \quad A_n(z^2) \cos z = \sum_{s=0}^{\infty} d_s z^{2s},$$

from which we get

$$(3.8) \quad \begin{aligned} c_s &= (-1)^s \sum_{k=0}^m (-1)^k b_k / (2s - 2k + 1)!, \\ d_s &= (-1)^s \sum_{k=0}^n (-1)^k a_k / (2s - 2k)!, \quad s = 0, 1, 2, \dots \end{aligned}$$

Substitute (3.6), (3.7) into (3.4) to get

$$(3.9) \quad f(z) = \sum_{s=0}^{\infty} c_s z^{2s+1} - \sum_{s=0}^{\infty} d_s z^{2s+r},$$

and by (2.6) we can write

$$(3.10) \quad \begin{aligned} f(z) &= C_0 z^q \prod_{k=1}^q (1 - z^2/\alpha_k^2), \\ \text{where } C_0 &= c_0, q = 1 \text{ for } r > 1, \\ C_0 &= c_i - d_i, q = 2i + 1 \text{ for } r = 1, c_i - d_i = 0, \\ & \quad j = 0, 1, \dots, i - 1, \text{ and } c_i - d_i \neq 0. \end{aligned}$$

Take the logarithmic derivatives of (3.9), (3.10), equate, and multiply by  $zf(z)$ ,  $f(z)$  as in (3.9) to get

$$(3.11) \quad \begin{aligned} \sum_{s=0}^{\infty} (2s + 1)c_s z^{2s+1} - \sum_{s=0}^{\infty} d_s (2s + r)z^{2s+r} \\ = \left[ q - 2 \sum_{s=0}^{\infty} S_s z^{2s+2} \right] \left[ \sum_{s=0}^{\infty} c_s z^{2s+1} - \sum_{s=0}^{\infty} d_s z^{2s+r} \right], \end{aligned}$$

or, after rearranging, and dividing by 2,

$$(3.12) \quad \begin{aligned} \sum_{s=0}^{\infty} z^{2s+3} \sum_{k=0}^s S_k [c_{s-k} - d_{s-i-k}] &= - \sum_{s=0}^{\infty} [s + 1 - (q - 1)/2] c_{s+1} z^{2s+3} \\ &+ \sum_{s=0}^{\infty} [s + 1 - (q - r)/2] d_{s+1} z^{2s+2+r}, \end{aligned}$$

with  $d_i = 0$  for  $j < 0$ . The coefficient of  $z$  on the right-hand side is

$$(q - 1)[c_0 - d_0 \delta_{r,1}]/2 = 0 \quad \text{by (3.10).}$$

Equate coefficients of  $z^{2l+3}$  in (3.12) to get

$$(3.13) \quad \begin{aligned} \sum_{k=0}^l S_k [c_{l-k} - d_{l-i-k}] &= -[l + 1 - (q - 1)/2][c_{l+1} - d_{l+1-i}], \\ & \quad l = 0, 1, 2, \dots \end{aligned}$$

Case 1.1:  $t > 0$ . In this case  $q = 1$  and (3.13) becomes

$$(3.14) \quad \sum_{k=0}^l S_k [c_{l-k} - d_{l-i-k}] = -(l + 1)[c_{l+1} - d_{l+1-i}], \quad l = 0, 1, 2, \dots,$$

from which we obtain, as special cases

$$(3.15) \quad \begin{aligned} S_0 &= \frac{1}{6} - b_1 + a_0 \delta_{t,1}, \\ S_1 &= b_1^2 - 2b_2 + \frac{1}{6} + [2a_1 - a_0(\frac{2}{3} + 2b_1 - a_0)]\delta_{t,1} + 2a_0 \delta_{t,2}. \end{aligned}$$

**THEOREM 2.** Let  $b_k = 0, k = 1, 2, \dots, p - 1$ , and  $b_p \neq 0$ . Define

$$(3.16) \quad S_l = \pi^{-2l-2} \zeta(2l + 2) + F_l, \quad l = 0, 1, 2, \dots,$$

where  $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$  is the Riemann zeta function. Then

$$(3.17) \quad F_l = 0, \quad l \leq \text{Min}(t - 2, p - 2),$$

$$(3.18) \quad F_l = (l + 1)[a_0 \delta_{l,t-1} - b_p \delta_{l,p-1}], \quad l = \text{Min}(t - 1, p - 1).$$

*Proof.* If  $l + 1 < t$  then (3.14) becomes

$$(3.19) \quad \sum_{k=0}^l S_k c_{l-k} = -(l + 1)c_{l+1}.$$

If also  $l + 1 < p$ , we get from (3.8) that for  $s \leq l + 1$ ,

$$c_s = (-1)^s \frac{b_0}{(2s + 1)!} = \frac{(-1)^s}{(2s + 1)!}.$$

So that if  $l \leq \text{Min}(t - 2, p - 2)$ , then (3.14) takes the form

$$(3.20) \quad \sum_{k=0}^l \frac{(-1)^k S_k}{(2l - 2k + 1)!} = \frac{l + 1}{(2l + 3)!},$$

which is equation [1, (21)], for the parameter  $k = 0$ . But in this case [1, (24)], we have  $S_l = \pi^{-2l-2} \zeta(2l + 2)$ , i.e., (3.17). If  $l = \text{Min}(t - 1, p - 1)$ , then (3.14) is

$$\sum_{k=0}^l S_k c_{l-k} = -(l + 1)[c_{l+1} - d_0 \delta_{l,t-1}],$$

or

$$\sum_{k=0}^l \frac{(-1)^{l-k} S_k}{(2l - 2k + 1)!} = (l + 1) \left[ \frac{(-1)^l}{(2l + 3)!} - b_p \delta_{l,p-1} + a_0 \delta_{l,t-1} \right].$$

(3.18) now follows from (3.16), (3.17) and (3.20).

Case 1.2:  $t = 0$ . In this case,  $i = (q - 1)/2$  (see (3.10)), and (3.13) is

$$(3.21) \quad \sum_{k=0}^l S_k [c_{l-k} - d_{l-k}] = -[l + 1 - i][c_{l+1} - d_{l+1}].$$

From (3.10),  $c_j - d_j = 0$  for  $j < i$ , and (3.21) is an identity,  $0 = 0$ , if  $l < i$ . For  $l \geq i$ , we have

$$\sum_{k=0}^{l-i} S_k (c_{l-k} - d_{l-k}) = -(l + 1 - i)[c_{l+1} - d_{l+1}],$$

or, replace  $l - i$  by  $l$ , to get

$$(3.22) \quad \sum_{k=0}^l S_k [c_{l+i-k} - d_{l+i-k}] = -(l + 1)(c_{l+i+1} - d_{l+i+1}),$$

$$l = 0, 1, 2, \dots$$

As special cases, we have

$$(3.23) \quad S_0 = -\frac{c_{i+1} - d_{i+1}}{c_i - d_i}, \quad S_1 = \left[ \frac{c_{i+1} - d_{i+1}}{c_i - d_i} \right]^2 - 2 \frac{c_{i+2} - d_{i+2}}{c_i - d_i}.$$

A theorem similar to Theorem 2, for this case will be given in Section 4.

Case 2. Consider now  $f(z)$  as in (3.5). From (3.6), (3.7), (3.8), we get

$$(3.24) \quad f(z) = \sum_{s=0}^{\infty} c_s z^{2s+1+r} - \sum_{s=0}^{\infty} d_s z^{2s+1},$$

and by (2.6), we write

$$(3.25) \quad f(z) = C_0 z^q \prod_{k=1}^{\infty} (1 - z^2/\alpha_k^2)$$

with  $C_0 = -a_0, q = 0$ . Repeat the same process as before to get

$$\begin{aligned} \sum_{s=0}^{\infty} z^{2s+1} \sum_{k=0}^s S_k d_{s-k} - \sum_{s=0}^{\infty} z^{2s+2+r} \sum_{k=0}^s S_k c_{s-k} \\ = \sum_{s=0}^{\infty} (s+t+1)c_s z^{2s+r} - \sum_{s=0}^{\infty} (s+1)d_{s+1} z^{2s+1} \end{aligned}$$

which yields the recurrence relation

$$(3.26) \quad \sum_{k=0}^l S_k [c_{l-t-1-k} - d_{l-k}] = -(l+1)[c_{l-t} - d_{l+1}], \quad l = 0, 1, 2, \dots$$

As special cases we get

$$(3.27) \quad \begin{aligned} S_0 &= \frac{1}{2} - \frac{a_1}{a_0} + \frac{1}{a_0} \delta_{l,0}, \\ S_1 &= \frac{1}{6} + \frac{a_1^2}{a_0^2} - \frac{2a_2}{a_0} + \left[ \frac{2}{3a_0} + \frac{1-2a_1}{a_0^2} + \frac{2b_1}{a_0} \right] \delta_{l,0} + 2\delta_{l,1}. \end{aligned}$$

The problem in [1, Section 6] is a special case of this, with  $t = 0, a_0 = -k, a_l = b_l = 0$  for  $l > 0$ . In this case, we get from (3.26), (3.8) and  $b_0 = 1,$

$$(3.28) \quad \sum_{s=0}^l \frac{(-1)^s (2l - 2s - k)}{(2l - 2s)!} S_s = \frac{2l + 2 - k}{2[(2l + 1)!]},$$

which coincides with [1, (36)] for  $S_s \equiv T_s(k), s = 0, 1, 2, \dots$

**4. Bessel Functions of Order One-Half.** The Bessel function  $J_{l+1/2}(z), l > 0,$  is given by [3, p. 298],

$$(4.1) \quad J_{l+1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} [R_{l,1/2}(z)\sin z - R_{l-1,3/2}(z)\cos z],$$

where [3, p. 296],

$$(4.2) \quad R_{l,\nu}(z) = \sum_{n=0}^{[l/2]} (-1)^n \binom{l-n}{n} \frac{\Gamma(\nu+l-n)}{\Gamma(\nu+n)} \left(\frac{1}{2}z\right)^{-l+2n}.$$

Also, [3, p. 40],

$$(4.3) \quad J_{l+1/2}(z) = \left(\frac{1}{2}z\right)^{l+1/2} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k! \Gamma(l+k+3/2)} = z^{l+1/2} P(z), \quad \text{where } P(0) \neq 0.$$

By (4.1), (4.3) we get

$$(4.4) \quad (\pi/2)^{1/2} z^{2l+1} P(z) = [z^l R_{l,1/2}(z)] \sin z - [z^{l-1} R_{l-1,3/2}(z)] z \cos z.$$

Now,  $z^l R_{l,1/2}(z) = 2^l g_{l,-1/2}(\frac{1}{4}z^2)$ , and  $z^{l-1} R_{l-1,3/2}(z) = 2^{l-1} g_{l-1,1/2}(\frac{1}{4}z^2)$ , where

$$(4.5) \quad g_{l,\nu}(\zeta) = \sum_{k=0}^{\lfloor l/2 \rfloor} \binom{l-k}{k} \frac{\Gamma(\nu+l-k+1)}{\Gamma(\nu+k+1)} \zeta^k$$

are the modified Lommel polynomials [3, p. 303].

Since

$$2^l g_{l,-1/2}(0) = \frac{2^l \Gamma(l + \frac{1}{2})}{\Gamma(\frac{1}{2})} \quad \text{and} \quad 2^{l-1} g_{l-1,1/2}(0) = \frac{2^{l-1} \Gamma(l + \frac{1}{2})}{\Gamma(\frac{3}{2})} = 2^l g_{l,-1/2}(0),$$

we get that

$$(4.6) \quad \begin{aligned} f_l(z) &= \frac{\Gamma(\frac{1}{2})}{2^l \Gamma(l + \frac{1}{2})} \left(\frac{\pi}{2}\right)^{1/2} z^{2l+1} P(z) \\ &= \frac{\Gamma(\frac{1}{2})}{2^l \Gamma(l + \frac{1}{2})} \left(\frac{\pi}{2}\right)^{1/2} z^{l+1/2} J_{l+1/2}(z) \\ &= B_m(z^2) \sin z - z A_n(z^2) \cos z, \end{aligned}$$

is of the form (3.4), with  $r = 1, b_0 = a_0 = 1$ ,

$$(4.7) \quad B_m(z^2) = \frac{\Gamma(\frac{1}{2})}{\Gamma(l + \frac{1}{2})} g_{l,-1/2}(\frac{1}{4}z^2)$$

and

$$(4.8) \quad A_n(z^2) = \frac{\Gamma(\frac{1}{2})}{\Gamma(l + \frac{1}{2})} g_{l-1,1/2}(\frac{1}{4}z^2),$$

where

$$(4.9) \quad n = m = (l - 1)/2, \quad l \text{ odd}, \quad n = l/2, \quad n = m - 1, \quad l \text{ even}.$$

We can therefore apply the results of Section 3 to  $J_{l+1/2}(z), l > 0$ . It is well known [3, p. 482] that the Bessel function  $J_\nu(z)$  has only real zeros for  $\nu > -1$ , and this is therefore true for  $J_{l+1/2}(z), l > 0$ . The following theorem is therefore of interest.

**THEOREM 3.** *If in  $J_{l+1/2}(z)$ , for  $l > 2$ , we perturb any one of the coefficients in  $R_{l,1/2}(z)$  or  $R_{l-1,3/2}(z)$ , the resultant function has nonreal zeros.*

To prove this theorem, we shall first prove two lemmas which are of interest themselves.

**LEMMA 3.** *The coefficients  $b_0, b_1, \dots, b_m, a_0, a_1, \dots, a_n$ , of (4.7), (4.8) are uniquely determined by the following set of linear nonhomogeneous equations*

$$(4.10) \quad b_0 = 1, \quad c_j = d_j, \quad j = 0, 1, \dots, l - 1,$$

where  $c_j, d_j$  are defined as in (3.8).

*Proof.*  $b_0 = 1$  is a condition of (3.4), which is satisfied by  $f_l(z)$  in (4.6). From (3.10) we have

$$(4.11) \quad f_l(z) = C_0 z^q \prod_{k=1}^{\infty} (1 - z^2/\alpha_k^2) = z^q P_1(z),$$

where  $P_1(0) \neq 0$ , and  $q = 2i + 1$  if  $c_i - d_i = 0$  for  $j = 0, 1, \dots, i - 1$ , and  $c_i - d_i \neq 0$ . From (4.6),  $f_i(z) = z^{2i+1}P_2(z)$ , with  $P_2(0) \neq 0$ . It follows that  $q = 2l + 1$ , and therefore  $i = l$ , and (4.10) is satisfied. The number of equations in (4.10) is  $l + 1$ , and the number of unknowns (counting  $b_0$ ) is  $n + m + 2$  which by (4.9) is equal to  $l + 1$ , so that (4.10) is a set of  $l + 1$  nonhomogenous linear equations in  $l + 1$  unknowns. For uniqueness, see Appendix.

LEMMA 4. Let  $\bar{b}_0 (= 1), \bar{b}_1, \dots, \bar{b}_m, \bar{a}_0, \bar{a}_1, \dots, \bar{a}_n$ , be the (unique) solution of (4.10). Let  $b_s = \bar{b}_s + \epsilon, a_s = \bar{a}_s + \delta, 0 \leq s \leq m$ , and  $b_j = \bar{b}_j, a_j = \bar{a}_j$  for  $j \neq s$ . [If  $n = m - 1$ , see (4.9), then  $\bar{a}_m = \delta = 0$ .] Let

$$(4.12) \quad f_i(z, s, \epsilon, \delta) = \frac{1}{b_0} \left[ \sum_{k=0}^m b_k z^{2k} \sin z - z \sum_{k=0}^n a_k z^{2k} \cos z \right],$$

and let  $S_k^l(s, \epsilon, \delta)$  denote the sums of the nonzero zeros (1.1) of  $f_i(z, s, \epsilon, \delta)$ . Then

$$(4.13) \quad S_j^l(s, \epsilon, 0) = \pi^{-2j-2} \zeta(2j + 2), \quad j \leq l - s - 2,$$

and

$$(4.14) \quad S_j^l(s, 0, \delta) = \pi^{-2j-2} (2^{2i+2} - 1) \zeta(2j + 2), \quad j \leq l - s - 2.$$

In particular, when  $j = 0$ , (4.13) holds for  $l > 2$ , and  $l = 2, s = 0$ , and (4.14) holds for  $l \geq 2$ .

Proof.  $f_i(z, s, \epsilon, \delta)$  is of the form (3.4), with  $r = 1$ . It follows from (3.8) and (4.10), that  $c_i - d_i = 0$  for  $j < s$ , and

$$(4.15) \quad \begin{aligned} c_s - d_s &= (-1)^s \left[ \sum_{k=0}^s \frac{(-1)^k \bar{b}_k}{(2s - 2k + 1)!} + (-1)^s \epsilon - \sum_{k=0}^s \frac{(-1)^k \bar{a}_k}{(2s - 2k)!} - (-1)^s \delta \right] \\ &= \epsilon - \delta \quad (= \epsilon, \text{ if } s = m = n + 1). \end{aligned}$$

For  $\epsilon \neq \delta$ , we can write, by (3.10),

$$(4.16) \quad f_i(z, s, \epsilon, \delta) = (\epsilon - \delta) z^{2s+1} \prod_{k=1}^{\infty} (1 - z^2/\alpha_k^2),$$

and by (3.22) we have

$$(4.17) \quad \sum_{k=0}^j S_k^l(s, \epsilon, \delta) [c_{s+i-k} - d_{s+i-k}] = -(j + 1) [c_{s+i+1} - d_{s+i+1}],$$

$j = 0, 1, 2, \dots$

If  $s + t \leq l - 1$ , then by (4.10) and (3.8),

$$(4.18) \quad c_{s+t} - d_{s+t} = \frac{(-1)^t}{(2t + 1)!} \cdot \epsilon - \frac{(-1)^t}{(2t)!} \cdot \delta,$$

and so for  $s + j + 1 \leq l - 1$ , (4.17) becomes

$$(4.19) \quad \sum_{k=0}^j S_k^l(s, \epsilon, \delta) (-1)^k \left[ \frac{\epsilon - (2j - 2k + 1)\delta}{(2j - 2k + 1)!} \right] = (j + 1) \frac{\epsilon - (2j + 3)\delta}{(2j + 3)!},$$

which is equation [1, (21)] with the parameter  $k = \delta/\epsilon \neq 1$ . In particular, if  $\epsilon \neq 0, \delta = 0$ , this is [1, (21)] with  $k = 0$ , and so by [1, (24)], we have (4.13). If  $\delta \neq 0, \epsilon = 0$ , this is [1, (21)] with  $k \rightarrow \infty$ , and so by [1, (28)] we have (4.14). Since  $s \leq m$ , it follows



from (4.9) that when  $j = 0$ , (4.13) holds for  $l > 2$  and  $l = 2, s = 0$ , and since  $s \leq n$  in (4.14), (4.14) holds for  $l \geq 2$ , if  $j = 0$ .

*Proof of Theorem 3.* The nonzero zeros of  $J_{l+1/2}(z)$  are the same as those of  $f_l(z)$  in (4.6), and perturbing a coefficient of  $R_{l,1/2}(z)$  or  $R_{l-1,s/2}(z)$  is equivalent to perturbing a coefficient of  $B_m(z^2)$  or  $A_n(z^2)$  in (4.6). We can therefore look at the function  $f_l(z)$ . Let  $b_j (j = 1, 2, \dots, m), a_j (j = 1, 2, \dots, n)$  be defined as in Lemma 4. Since  $l > 2$ , we have by (4.13)

$$(4.20) \quad S_0^l(s, \epsilon, 0) = \pi^{-2} \zeta(2) = \frac{1}{6}, \quad \epsilon \neq 0,$$

and by (4.14),

$$(4.21) \quad S_0^l(s, 0, \delta) = \pi^{-2} 3\zeta(2) = \frac{1}{2}, \quad \delta \neq 0.$$

From (4.20) it follows that

$$(4.22) \quad \lim_{\epsilon \rightarrow 0} S_0^l(s, \epsilon, 0) = \frac{1}{6}.$$

The roots  $\pm\alpha_1, \pm\alpha_2, \dots$  depend continuously on  $\epsilon$ . If  $\epsilon \neq 0 (\delta = 0)$ , we have from (4.16) that the multiplicity at zero is of order  $2s + 1$ , but when  $\epsilon = 0 (\delta = 0)$ , we have from (4.6) that the multiplicity at zero is  $2l + 1$ . Since the roots occur in pairs  $\pm\alpha$ , we must have

$$(4.23) \quad \lim_{\epsilon \rightarrow 0} \alpha_j^2 = 0, \quad j = 1, 2, \dots, l - s.$$

But

$$\frac{1}{6} = S_0^l(s, \epsilon, 0) = \sum_{k=1}^{l-s} \alpha_k^{-2} + \sum_{k=l-s+1}^{\infty} \alpha_k^{-2},$$

and

$$\lim_{\epsilon \rightarrow 0} \sum_{k=l-s+1}^{\infty} \alpha_k^{-2} = \frac{1}{2(2l+3)},$$

see [3, p. 502] or [4]. We therefore get that  $Q(\epsilon) = \sum_{k=1}^{l-s} \alpha_k^{-2}$  stays finite as  $\epsilon \rightarrow 0$ : If all  $\alpha_k^2$  in  $Q(\epsilon)$  tended to zero through positive values, then  $\lim_{\epsilon \rightarrow 0} Q(\epsilon) = \infty$ . So that, when  $\epsilon \neq 0 (|\epsilon| \ll 1)$ , we must have at least one root,  $\alpha_j$ , for which  $\alpha_j^2$  is not positive, i.e.,  $\alpha_j$  is nonreal. The same argument applied to  $S_0^l(s, 0, \delta)$  by (4.21) again yields that if  $\delta \neq 0 (|\delta| \ll 1)$  we must have nonreal roots, completing the proof.

*Remark.* From the proof and Lemma 4, it is obvious that Theorem 3 holds also when  $l = 2, s = 0$ . (For  $l = 2$ , we only have  $a_0, b_0$  and  $b_1$ , and the theorem does not apply to  $b_1$ .)

There are three cases which Theorem 3 does not cover, if  $l + \frac{1}{2} > 0$ ; the cases  $l = 0, l = 1$ , and  $l = 2, s = 1$  (only  $b_1$ ). The case  $l = 0$  is not of the form we are discussing. Indeed  $(\pi z/2)^{1/2} J_{1/2}(z) = \sin z$ , so we only have  $b_0 = 1$ , and obviously the roots do not depend on changing  $b_0$  to any nonzero constant. The other two cases are indeed different from the cases covered by Theorem 3, and we have the following.

**THEOREM 4.** *Under the same definitions as in Lemma 4,*

(1)  $f_l(z, 0, -\epsilon, 0)$  and  $f_l(z, 0, 0, \delta)$  have only real zeros for  $\epsilon \geq 0, \delta \geq 0$ , and a pair of imaginary roots for  $\epsilon < 0, \delta < 0 (|\epsilon| \ll 1, |\delta| \ll 1)$ .

(2)  $f_2(z, 1, -\epsilon, 0)$  has only real roots for  $\epsilon \geq 0$ , and a pair of imaginary roots for  $\epsilon < 0$  ( $|\epsilon| \ll 1$ ).

*Proof.* *Part (1).* This is the problem of [1, Section 1], with the parameter  $k = a_0/b_0$  there, and (1) follows.

*Part (2).* Note that

$$(4.24) \quad f_2(z) = (1 - \frac{1}{3}z^2) \sin z - z \cos z.$$

If we look at the function  $g(z) = (k_1 - k_2z^2) \sin z - z \cos z$ , the roots,  $\pm\alpha$  of  $g(z)$  arise from the Sturm-Liouville system

$$(4.25) \quad \begin{aligned} u'' + \alpha^2 u &= 0, \\ Cu(0) + Du'(0) &= 0, \\ Eu(1) - Fu'(1) &= 0, \quad ED + CF \neq 0, \end{aligned}$$

where  $k_1 = CE/(ED + CF)$ ,  $k_2 = FD/(ED + CF)$ .

By Sturm-Liouville theory,  $\alpha^2$  are all real, and therefore  $f_2(z, 1, -\epsilon, 0)$  may have either real or imaginary roots. An imaginary root  $z = \pm ix$ ,  $x > 0$ , of  $f_2(z, 1, -\epsilon, 0)$  should satisfy the equation

$$(4.26) \quad y_1(x) = y_2(x),$$

where  $y_1(x) = \tanh x$ ,  $y_2(x) = x/(1 + cx^2)$ ,  $c = \frac{1}{3} + \epsilon$ . Since  $l - s = 2 - 1 = 1$  only one such  $x > 0$  can exist (for  $\epsilon$  small enough).  $y_1(0) = y_2(0) = 0$  and  $\lim_{x \rightarrow \infty} y_1(x) = 1$ ,  $\lim_{x \rightarrow \infty} y_2(x) = 0$ . As  $y_1(x)$ ,  $y_2(x)$  may intersect only once, for  $x > 0$ , it is necessary and sufficient that  $y_2(x) > y_1(x)$  in some interval  $(0, \eta^2)$ , for the existence of a positive solution of (4.26). Checking derivatives at zero, one finds  $y_1^{(j)}(0) = y_2^{(j)}(0)$  for  $j = 0, 1, 2$ , and  $y_1^{(3)}(0) = -2$ ,  $y_2^{(3)}(0) = -6c$ , and therefore, there exists a positive root of (4.26) iff  $-6c > -2$ , i.e.,  $c < \frac{1}{3}$ . This concludes the proof of Part (2).

To get an idea of the asymptotic behavior of the roots which split away from zero when  $\epsilon$  (or  $\delta$ ) tend to zero, note that by the definitions,  $\lim_{\epsilon \rightarrow 0} f_i(z, s, \epsilon, 0) = \lim_{\delta \rightarrow 0} f_i(z, s, 0, \delta) = f_i(z)$ . From (4.16),

$$(4.27) \quad \begin{aligned} f_i(z, s, \epsilon, 0) &= \epsilon z^{2s+1} \prod_{k=1}^{\infty} (1 - z^2/\alpha_k^2) \\ &= z^{2l+1} \prod_{k=l-s+1}^{\infty} (1 - z^2/\alpha_k^2) \prod_{k=1}^{l-s} (1/z^2 - 1/\alpha_k^2) \epsilon \\ &= z^{2l+1} \prod_{k=l-s+1}^{\infty} (1 - z^2/\alpha_k^2) \prod_{k=1}^{l-s} (1 - \alpha_k^2/z^2) (-1)^{l-s} \epsilon \prod_{k=1}^{l-s} \alpha_k^{-2}. \end{aligned}$$

Take the limit  $\epsilon \rightarrow 0$  of (4.27) and use (4.23), (4.6), the continuous dependence of the roots on  $\epsilon$ , and (3.10), to get

$$f_i(z) = f_i(z) \lim_{\epsilon \rightarrow 0} (-1)^{l-s} \epsilon (c_l - d_l)^{-1} \prod_{k=1}^{l-s} \alpha_k^{-2},$$

or

$$\lim_{\epsilon \rightarrow 0} \prod_{k=1}^{l-s} \alpha_k^{-2} \cdot \epsilon = (-1)^{l-s} (c_l - d_l).$$

From (4.3), (4.6) we get that

$$(4.28) \quad c_l - d_l = (2l + 1)^{-1} [2^l l! / (2l)!]^2,$$

and we get

$$(4.29) \quad \lim_{\epsilon \rightarrow 0} \prod_{k=1}^{l-s} \alpha_k^{-2} \cdot \epsilon = (-1)^{l-s} (2l + 1)^{-1} [2^l l! / (2l)!]^2.$$

Again, when  $\epsilon = 0, \delta \neq 0$ , (4.29) holds with  $\epsilon$  replaced by  $(-\delta)$ .

*Example.*  $l = 3, s = 1, \epsilon < 0$ . By Theorem 3, we know we must have at least one nonreal root. By (4.29),  $\lim_{\epsilon \rightarrow 0} \epsilon \alpha_1^{-2} \alpha_2^{-2} = 1/1575$ , and from Theorem 3, we have  $\lim_{\epsilon \rightarrow 0} \alpha_1^{-2} + \alpha_2^{-2} = \frac{1}{3}$ . If  $|\epsilon| \ll 1$ , we must therefore have  $\alpha^2 = \alpha_1^2 > 0$ , and  $-\beta^2 = \alpha_2^2 < 0$ , and so the two double roots at zero separate to  $\pm\alpha$  ( $\alpha > 0$ ) along the real axis, and  $\pm i\beta$  ( $\beta > 0$ ) along the imaginary axis. For  $|\epsilon| \ll 1, \alpha^2 \beta^2 \approx 1575(-\epsilon), \beta^2 - \alpha^2 \approx 175(-\epsilon)$ , or  $\alpha^2 \approx \beta^2 \approx 40 |\epsilon|^{1/2} + O(\epsilon)$ .

We have [3, p. 298],

$$(4.30) \quad (-1)^l J_{-l-1/2}(z) = (2/\pi z)^{1/2} [R_{l-1,3/2}(z) \sin z + R_{l,1/2}(z) \cos z],$$

and so

$$\frac{\Gamma(\frac{1}{2})}{2^l \Gamma(l + \frac{1}{2})} (-1)^l \left(\frac{\pi}{2}\right)^{1/2} z^{l+1/2} J_{-l-1/2}(z), \quad l > 0,$$

is of the form (3.5) with  $r = 1, m = n = (l - 1)/2$  for  $l$  odd and  $n = l/2, m = n - 1$  for  $l$  even, and the coefficients are given by

$$(4.31) \quad b_j = \bar{a}_j, \quad a_j = -\bar{b}_j, \quad j = 1, 2, \dots, n,$$

where  $\bar{a}_j, \bar{b}_j$  ( $j = 1, \dots, n$ ) are the solutions of (4.10).

Since  $r = 1$ , (3.26) holds with  $t = 0$ , and from (3.27), (4.31), we get

$$(4.32) \quad S_0 = -\frac{1}{2} - \bar{b}_1, \quad S_1 = \frac{1}{2} + \bar{b}_1^2 + 2\bar{b}_2 + 2\bar{b}_1 - 2\bar{a}_1.$$

**Appendix.** We want to prove that there exists a unique solution  $b_0, \dots, b_m, a_0, \dots, a_n$  to the set of equations

$$(A.1) \quad c_s - d_s = (-1)^s \left[ \sum_{k=0}^m \frac{(-1)^k b_k}{(2s - 2k + 1)!} - \sum_{k=0}^n \frac{(-1)^k a_k}{(2s - 2k)!} \right] = 0,$$

$$s = 0, 1, 2, \dots, l - 1,$$

$m = n = (l - 1)/2$ , for  $l$  odd, and  $m = l/2, n = m - 1$  for  $l$  even.

Note that we can replace  $n$  by  $m$ , and add the equation  $a_m = 0$ , for  $l$  even. We know that for any  $l \geq 0$  there exists a solution to (A.1) given by the coefficients in (4.7), (4.8). From (3.9), (A.1) holds iff

$$(A.2) \quad f(z) = \sin z \sum_{k=0}^m b_k z^{2k} - z \cos z \sum_{k=0}^n a_k z^{2k}, \quad (b_0 = 1),$$

has a zero of multiplicity at least  $2l + 1$  at zero, and actually, for the solution we

know it is exactly of multiplicity  $2l + 1$ , by (4.28). Thus, (A.1) admits a unique solution iff any other choice of the coefficients,  $b_0 (= 1), b_1, \dots, b_m, a_0, a_1, \dots, a_n$ , would have the corresponding function  $f(z)$  in (A.2), with a zero of multiplicity less than  $2l + 1$ , at zero. We therefore rephrase what we want to prove as follows:

**THEOREM 5.** *There exists a unique set of coefficients  $b_0, \dots, b_m, a_0, \dots, a_m$  such that, at zero,  $f(z)$ , defined by (A.2), has a zero of multiplicity  $4m + 1$  ( $l = 2m$ ) if we demand  $a_m = 0$ , and multiplicity  $4m + 3$  ( $l = 2m + 1$ ) otherwise, and such that  $b_0 = 1$ .*

*Proof.* By induction.

(1)  $m = 0$ . From (A.1),  $b_0 = 1$  for  $l = 0$ , and  $a_0 = b_0 = 1$  for  $l = 1$ , and the solution is unique, so the theorem holds.

(2) The induction hypothesis is that the theorem is true for  $m = 0, 1, \dots, i$ . We want to prove it for  $m = i + 1$ . Note that the first two equations in (A.1) are always  $b_0 = 1, a_0 = b_0$ , and so

$$(A.3) \quad b_0 = a_0 = 1 \quad \text{for all } m > 0.$$

*Part a.* Let  $\bar{b}_0, \dots, \bar{b}_{i+1}, \bar{a}_0, \dots, \bar{a}_i, \bar{a}_{i+1} [=0]$ , and  $b_0, \dots, b_{i+1}, a_0, \dots, a_{i+1} [=0]$  be two distinct solutions of (A.1) for  $l = 2i + 2$ . It follows from (A.2) and (3.9) that we can write

$$(A.4) \quad f_1(z) = \sin z \sum_{k=0}^{i+1} \bar{b}_k z^{2k} - z \cos z \sum_{k=0}^i \bar{a}_k z^{2k} = z^{4i+5} R_1(z^2)$$

and

$$(A.5) \quad f_2(z) = \sin z \sum_{k=0}^{i+1} b_k z^{2k} - z \cos z \sum_{k=0}^i a_k z^{2k} = z^{4i+5} R_2(z^2).$$

Subtract (A.5) from (A.4) to get

$$(A.6) \quad \sin z \sum_{k=0}^{i+1} (\bar{b}_k - b_k) z^{2k} - z \cos z \sum_{k=0}^i (\bar{a}_k - a_k) z^{2k} = z^{4i+5} [R_1(z^2) - R_2(z^2)],$$

and, by (A.3), we must have

$$(A.7) \quad a_0 = \bar{a}_0 = b_0 = \bar{b}_0 = 1.$$

From (A.1) it follows that  $\bar{b}_k - b_k = 0$  for all  $k \leq j - 1 \leq m + 1$  iff  $\bar{a}_k - a_k = 0$  for all  $k \leq j - 1 \leq m + 1$ . Suppose  $\bar{b}_k - b_k = 0$  for  $k = 0, 1, 2, \dots, j - 1$ , and  $\bar{b}_j - b_j \neq 0$ .  $j \geq 1$  because of (A.7), and  $j \leq i$  since the two solutions are distinct, and  $\bar{a}_{i+1} = a_{i+1} = 0$ . Divide (A.6) by  $z^{2j}$  to get

$$(A.8) \quad \sin z \sum_{k=0}^{i+1-j} [\bar{b}_{k+j} - b_{k+j}] z^{2k} - z \cos z \sum_{k=0}^{i-j} [\bar{a}_{k+j} - a_{k+j}] z^{2k} \\ = z^{4i+5-2j} [R_1(z^2) - R_2(z^2)].$$

Divide through by  $\bar{b}_j - b_j (\neq 0)$ , and the left-hand side of (A.8) is of the form (A.2) with  $0 \leq m \leq i + 1 - j \leq i$ , and  $a_m = 0$ . From the induction hypothesis, it follows that the left-hand side of (A.8) can have a zero of multiplicity at most  $4m + 1 \leq 4i + 5 - 4j$  at zero. But the right-hand side of (A.8) has multiplicity of at least  $4i + 5 - 2j > 4i + 5 - 4j$ , a contradiction. Thus, (A.1) admits a unique solution for  $l = 2i + 2$ .

Part b. Let  $\bar{b}_0, \dots, \bar{b}_{i+1}, \bar{a}_0, \dots, \bar{a}_{i+1}$  and  $b_0, \dots, b_{i+1}, a_0, \dots, a_{i+1}$  be two distinct solutions of (A.1) for  $l = 2i + 3$ . Repeat the same process as in Part a to get the equation

$$(A.9) \quad \sin z \sum_{k=0}^{i+1-j} [\bar{b}_{k+j} - b_{k+j}]z^{2k} - \sum_{k=0}^{i+1-j} [\bar{a}_{k+j} - a_{k+j}]z^{2k} = z^{4i+7-2j}[R_3(z^2) - R_4(z^2)],$$

and  $1 \leq j \leq m + 1$ . Again divide by  $\bar{b}_j - b_j (\neq 0)$ , and get the left-hand side of (A.9) in the form (A.2) with  $0 \leq m \leq i + 1 - j \leq i$ . Since we do not require  $a_m = 0$ , we have by the induction hypothesis a zero of multiplicity at most  $4m + 3 \leq 4i + 7 - 4j$  for the left-hand side of (A.9) at zero. The right-hand side has, at zero, a zero of multiplicity at least  $4i + 7 - 2j > 4i + 7 - 4j$ , since  $j \geq 1$ , a contradiction. This concludes the proof.

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