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Infinite Sums of Roots for a Class of Transcendental Equations and Bessel Functions of Order One-Half

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Abstract. The roots of Bessel functions of order one-half are special cases of roots of transcendental equations of the form $\tan z = A(z)/B(z)$, where $A(z)$, $B(z)$ are polynomials and $A(z)/B(z)$ is odd. We prove that the function $f(z) = B(z)\sin z - A(z)\cos z$, $f(z)$ even or odd, satisfies the conditions of Hadamard's factorization theorem, and derive recurrences for sums of the form $S_l = \sum_{k=1}^{\infty} \alpha_k^{-2l}$, $l = 1, 2, \cdots$, where the α_k's are the nonzero roots of $f(z)$. We also prove under what conditions on $A(z)$ and $B(z)$ is $S_l = \pi^{-2l-2}\zeta(2l + 2)$ or $S_l = \pi^{-2l-2}\zeta(2l + 2)(2^{2l+2} - 1)$, where ζ is the Riemann zeta function. We prove that, although Bessel functions of positive half-order, $J_{l+1/2}$, have only real roots, perturbation of any one of its coefficients introduces nonreal roots for $l > 2$.

1. Introduction. We are interested in sums of the form

$$S_k = \sum_{n=1}^{\infty} \alpha_n^{-2k-2}, \tag{1.1}$$

where the α_n's are the nonzero roots of a function of the type

$$f(z) = B_m(z)\sin z - A_n(z)\cos z, \tag{1.2}$$

where $B_m(z)$, $A_n(z)$ are polynomials of order m, n, respectively, $m \neq n$, and $f(z)$ is either even or odd. Since the roots occur in pairs, $\pm\alpha$, we take only one of each pair.

Two special cases of (1.2), $B_m(z) = 1$, $A_n(z) = kz$, and $B_m(z) = z$, $A_n(z) = -k$, k a nonzero real constant, have been treated in [1] using Sturm-Liouville theory.

We shall show that (1.2) has a discrete sequence of roots, α_n, with $\alpha_n^2 \to \infty$. As a special case, we get the Bessel functions $J_{l+1/2}(z)$ and $J_{-l-1/2}(z)$, for $l > 0$. For $J_{l+1/2}(z)$ we prove, by using S_0, that if $l > 2$, although $J_{l+1/2}(z)$ itself has only real roots, perturbations of any one of its coefficients (written in the form (1.2)), introduce nonreal roots.

2. Main Theorem.

Lemma 1. *Let*

$$f(z) = B_m(z)\sin z - A_n(z)\cos z, \qquad m \neq n, \tag{2.1}$$

where $B_m(z)$, $A_n(z)$ are polynomials of order m, n, respectively, which have no common root.

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Let $z = x + iy$. Then

(1) *There exists a $Y > 0$, such that if $y > Y$, $f(z)$ has no roots.*

(2) *In any strip $|x| \leqq L < \infty$, there can only be a finite number of roots of $f(z)$.*

Proof. Part (1). $f(z) = 0$ iff

$$\tan z = A_n(z)/B_m(z), \tag{2.2}$$

(including both sides $= \infty$). For $|z| \gg 1$,

$$|A_n(z)/B_m(z)| \sim C|z|^{n-m} = C|z|^k, \tag{2.3}$$

where $k = n - m$ and $C > 0$. If $|z| \gg 1$, we must therefore have

$$|\tan z| \approx C|z|^k, \tag{2.4}$$

if z is a root. But

$$\tan z = \tan(x + iy) = \frac{\tan x + i \tanh y}{1 - i \tan x \tanh y}$$

$$= \frac{\sin x \cos x\,(1 - \tanh^2 y) + i \tanh y}{\cos^2 x + \sin^2 x \tanh^2 y},$$

from which it follows that

$$|\tanh y| \leqq |\tan z| \leqq \frac{1}{2} + \frac{1}{2\tanh^2 y}, \tag{2.5}$$

i.e., $\tan z$ is bounded away from zero and infinity, if $y \neq 0$. If $|y| \gg 1$, then (2.5) implies that $|\tan z| \approx 1$. But if $|y| \gg 1$, then $|z| \gg 1$, and (2.4) holds for a root, i.e., $|\tan z| \gg 1$ for $k > 0$, and $|\tan z| \ll 1$ for $k < 0$.

This concludes the proof of (1).

Part (2). Suppose we had an infinite number of roots for $|x| \leqq L < \infty$. By Part (1), they would be in a bounded domain, and would have an accumulation point other than infinity. Since $f(z)$ is an analytic entire function, it follows that $f(z) \equiv 0$, by Taylor's Theorem. This contradiction proves Part (2).

Remark. It follows from (2.4) that we do have an infinite sequence of roots, tending to $\pm\infty$ on the real line, with the asymptotic values $n\pi$ for $k < 0$ and $(n + \frac{1}{2})\pi$ for $k > 0$, n an integer.

LEMMA 2. *Let $\lambda \equiv$ order of $f(z)$. Then $\lambda \leqq 1$.*

Proof. Let $M(r) = \operatorname{Max}_{|z|=r} |f(z)|$. Then, $M(r) \leqq (n + m)C_1 r^{m+n} e^r$, for $r > 1$, and C_1 is the largest coefficient in absolute value of $A_n(z)$ and $B_m(z)$.

$$\log M(r) \leqq \log[(n + m)C_1] + (m + n)\log r + r$$

$$\leqq C_2 r, \quad \text{for } r \text{ large enough and } C_2 > 0.$$

So that, $\log\log M(r) \leqq \log C_2 + \log r$, and therefore,

$$\lambda = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} \leqq 1 + \lim_{r\to\infty} \frac{\log C_2}{\log r} = 1.$$

THEOREM 1. *Let $f(z) = B_m(z)\sin z - A_n(z)\cos z$, where $B_m(z)$, $A_n(z)$, are polynomials of order m, n, respectively, one being even and the other odd. Then*

$$f(z) = C_0 z^q \prod_{k=1}^{\infty} (1 - z^2/\alpha_k^2), \tag{2.6}$$

where α_k are the roots of $f(z) = 0$, such that $|\alpha_1| \leqq |\alpha_2| \leqq \cdots \leqq |\alpha_n| \leqq \cdots$, q is the multiplicity of the root $z = 0$, and $C_0 = f^{(q)}(0)/q!$.

Proof. From Lemma 1 and the following remark we get that there exists an infinite sequence of roots of $f(z) = 0$, accumulating only at infinity. Also, the roots occur in pairs $\pm\alpha$. By Hadamard's factorization theorem [2, p. 22], we can write

$$f(z) = C_0 \exp\{g(z)\} z^q \prod_{k=1}^{\infty} (1 - z^2/\alpha_k^2), \qquad C_0 \neq 0, \tag{2.7}$$

where $q!\, C_0 = f^{(q)}(0)$, and $g(z)$ is a polynomial in z of order $\leqq \lambda$, with $g(0) = 0$. By Lemma 2, $\lambda \leqq 1$, and we get

$$g(z) = az. \tag{2.8}$$

Since $f(z)$ is even or odd, we get that q must be even or odd with $f(z)$, and $\exp\{g(z)\} = \exp\{g(-z)\}$, i.e., $az = -az$ by (2.8), or $a = 0$, which proves (2.6).

3. Recurrences and Special Cases of S_k. Rewrite $f(z)$ in Theorem 1 as

$$f(z) = b_{l+2m}(z) \sin z - a_{p+2n}(z) \cos z, \tag{3.1}$$

where

$$b_{l+2m}(z) = z^l \sum_{k=0}^{m} b_k z^{2k} = z^l B_m(z^2), \qquad b_0 \neq 0,\ b_m \neq 0, \tag{3.2}$$

and

$$a_{p+2n}(z) = z^p \sum_{k=0}^{n} a_k z^{2k} = z^p A_n(z^2), \qquad a_0 \neq 0,\ a_n \neq 0, \tag{3.3}$$

and $l + p$ is an odd integer.

Since we are interested in the nonzero roots of $f(z)$, we can divide by $b_0 z^{\mathrm{Min}(l,p)}$. The two basic cases are therefore:

$$f(z) = B_m(z^2) \sin z - z^r A_n(z^2) \cos z, \qquad r = 2t + 1 > 0, \tag{3.4}$$

and

$$f(z) = z^r B_m(z^2) \sin z - A_n(z^2) \cos z, \qquad r = 2t + 1 > 0, \tag{3.5}$$

with $b_0 = 1$ in both cases.

Case 1. Consider $f(z)$ as in (3.4). Write

$$B_m(z^2) \sin z = \sum_{s=0}^{\infty} c_s z^{2s+1}, \tag{3.6}$$

$$A_n(z^2) \cos z = \sum_{s=0}^{\infty} d_s z^{2s}, \tag{3.7}$$

from which we get

$$c_s = (-1)^s \sum_{k=0}^{m} (-1)^k b_k/(2s - 2k + 1)!,$$
(3.8)
$$d_s = (-1)^s \sum_{k=0}^{n} (-1)^k a_k/(2s - 2k)!, \qquad s = 0, 1, 2, \cdots$$

Substitute (3.6), (3.7) into (3.4) to get

$$f(z) = \sum_{s=0}^{\infty} c_s z^{2s+1} - \sum_{s=0}^{\infty} d_s z^{2s+r}, \tag{3.9}$$

and by (2.6) we can write

$$f(z) = C_0 z^q \prod_{k=1}^{\infty} (1 - z^2/\alpha_k^2),$$
(3.10) where $C_0 = c_0$, $q = 1$ for $r > 1$,
$$C_0 = c_i - d_i,\ q = 2i + 1 \text{ for } r = 1, c_j - d_j = 0,$$
$$j = 0, 1, \cdots, i - 1, \text{ and } c_i - d_i \neq 0.$$

Take the logarithmic derivatives of (3.9), (3.10), equate, and multiply by $zf(z)$, $f(z)$ as in (3.9) to get

$$\sum_{s=0}^{\infty} (2s + 1)c_s z^{2s+1} - \sum_{s=0}^{\infty} d_s(2s + r)z^{2s+r}$$
(3.11)
$$= \left[q - 2 \sum_{s=0}^{\infty} S_s z^{2s+2}\right]\left[\sum_{s=0}^{\infty} c_s z^{2s+1} - \sum_{s=0}^{\infty} d_s z^{2s+r}\right],$$

or, after rearranging, and dividing by 2,

$$\sum_{s=0}^{\infty} z^{2s+3} \sum_{k=0}^{s} S_k[c_{s-k} - d_{s-t-k}] = -\sum_{s=0}^{\infty} [s + 1 - (q - 1)/2]c_{s+1} z^{2s+3}$$
(3.12)
$$+ \sum_{s=0}^{\infty} [s + 1 - (q - r)/2]d_{s+1} z^{2s+2+r},$$

with $d_j = 0$ for $j < 0$. The coefficient of z on the right-hand side is

$$(q - 1)[c_0 - d_0\delta_{r,1}]/2 = 0 \quad \text{by (3.10)}.$$

Equate coefficients of z^{2l+3} in (3.12) to get

$$\sum_{k=0}^{l} S_k[c_{l-k} - d_{l-t-k}] = -[l + 1 - (q - 1)/2][c_{l+1} - d_{l+1-t}], \tag{3.13}$$
$$l = 0, 1, 2, \cdots.$$

Case 1.1: $t > 0$. In this case $q = 1$ and (3.13) becomes

$$\sum_{k=0}^{l} S_k[c_{l-k} - d_{l-t-k}] = -(l + 1)[c_{l+1} - d_{l+1-t}], \qquad l = 0, 1, 2, \cdots, \tag{3.14}$$

from which we obtain, as special cases

(3.15)
$$S_0 = \tfrac{1}{6} - b_1 + a_0\delta_{t,1},$$
$$S_1 = b_1^2 - 2b_2 + \tfrac{1}{90} + [2a_1 - a_0(\tfrac{2}{3} + 2b_1 - a_0)]\delta_{t,1} + 2a_0\delta_{t,2}.$$

THEOREM 2. *Let* $b_k = 0$, $k = 1, 2, \cdots, p-1$, *and* $b_p \neq 0$. *Define*

$$S_l = \pi^{-2l-2}\zeta(2l+2) + F_l, \qquad l = 0, 1, 2, \cdots, \tag{3.16}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ *is the Riemann zeta function. Then*

$$F_l = 0, \qquad l \leqq \operatorname{Min}(t-2, p-2), \tag{3.17}$$

$$F_l = (l+1)[a_0\delta_{l,t-1} - b_p\delta_{l,p-1}], \qquad l = \operatorname{Min}(t-1, p-1). \tag{3.18}$$

Proof. If $l + 1 < t$ then (3.14) becomes

$$\sum_{k=0}^{l} S_k c_{l-k} = -(l+1)c_{l+1}. \tag{3.19}$$

If also $l + 1 < p$, we get from (3.8) that for $s \leqq l + 1$,

$$c_s = (-1)^s \frac{b_0}{(2s+1)!} = \frac{(-1)^s}{(2s+1)!}.$$

So that if $l \leqq \operatorname{Min}(t-2, p-2)$, then (3.14) takes the form

$$\sum_{k=0}^{l} \frac{(-1)^k S_k}{(2l-2k+1)!} = \frac{l+1}{(2l+3)!}, \tag{3.20}$$

which is equation [1, (21)], for the parameter $k = 0$. But in this case [1, (24)], we have $S_l = \pi^{-2l-2}\zeta(2l+2)$, i.e., (3.17). If $l = \operatorname{Min}(t-1, p-1)$, then (3.14) is

$$\sum_{k=0}^{l} S_k c_{l-k} = -(l+1)[c_{l+1} - d_0\delta_{l,t-1}],$$

or

$$\sum_{k=0}^{l} \frac{(-1)^{l-k} S_k}{(2l-2k+1)!} = (l+1)\left[\frac{(-1)^l}{(2l+3)!} - b_p\delta_{l,p-1} + a_0\delta_{l,t-1}\right].$$

(3.18) now follows from (3.16), (3.17) and (3.20).

Case 1.2: $t = 0$. In this case, $i = (q-1)/2$ (see (3.10)), and (3.13) is

$$\sum_{k=0}^{l} S_k[c_{l-k} - d_{l-k}] = -[l+1-i][c_{l+1} - d_{l+1}]. \tag{3.21}$$

From (3.10), $c_j - d_j = 0$ for $j < i$, and (3.21) is an identity, $0 = 0$, if $l < i$. For $l \geqq i$, we have

$$\sum_{k=0}^{l-i} S_k(c_{l-k} - d_{l-k}) = -(l+1-i)[c_{l+1} - d_{l+1}],$$

or, replace $l - i$ by l, to get

$$\sum_{k=0}^{l} S_k[c_{l+i-k} - d_{l+i-k}] = -(l+1)(c_{l+i+1} - d_{l+i+1}), \qquad l = 0, 1, 2, \cdots. \tag{3.22}$$

As special cases, we have

$$S_0 = -\frac{c_{i+1} - d_{i+1}}{c_i - d_i}, \qquad S_1 = \left[\frac{c_{i+1} - d_{i+1}}{c_i - d_i}\right]^2 - 2\frac{c_{i+2} - a_{i+2}}{c_i - d_i}. \tag{3.23}$$

A theorem similar to Theorem 2, for this case will be given in Section 4.

Case 2. Consider now $f(z)$ as in (3.5). From (3.6), (3.7), (3.8), we get

$$f(z) = \sum_{s=0}^{\infty} c_s z^{2s+1+r} - \sum_{s=0}^{\infty} d_s z^{2s+1}, \tag{3.24}$$

and by (2.6), we write

$$f(z) = C_0 z^q \prod_{k=1}^{\infty} (1 - z^2/\alpha_k^2) \tag{3.25}$$

with $C_0 = -a_0$, $q = 0$. Repeat the same process as before to get

$$\sum_{s=0}^{\infty} z^{2s+1} \sum_{k=0}^{s} S_k d_{s-k} - \sum_{s=0}^{\infty} z^{2s+2+r} \sum_{k=0}^{s} S_k c_{s-k}$$
$$= \sum_{s=0}^{\infty} (s + t + 1) c_s z^{2s+r} - \sum_{s=0}^{\infty} (s + 1) d_{s+1} z^{2s+1}$$

which yields the recurrence relation

$$\sum_{k=0}^{l} S_k [c_{l-t-1-k} - d_{l-k}] = -(l + 1)[c_{l-t} - d_{l+1}], \qquad l = 0, 1, 2, \cdots . \tag{3.26}$$

As special cases we get

$$\begin{aligned} S_0 &= \frac{1}{2} - \frac{a_1}{a_0} + \frac{1}{a_0}\,\delta_{t,0}, \\ S_1 &= \frac{1}{6} + \frac{a_1^2}{a_0} - \frac{2a_2}{a_0} + \left[\frac{2}{3a_0} + \frac{1 - 2a_1}{a_0^2} + \frac{2b_1}{a_0}\right]\delta_{t,0} + 2\delta_{t,1}. \end{aligned} \tag{3.27}$$

The problem in [1, Section 6] is a special case of this, with $t = 0$, $a_0 = -k$, $a_l = b_l = 0$ for $l > 0$. In this case, we get from (3.26), (3.8) and $b_0 = 1$,

$$\sum_{s=0}^{l} \frac{(-1)^s(2l - 2s - k)}{(2l - 2s)!}\, S_s = \frac{2l + 2 - k}{2[(2l + 1)!]}, \tag{3.28}$$

which coincides with [1, (36)] for $S_s \equiv T_s(k)$, $s = 0, 1, 2, \cdots$.

4. Bessel Functions of Order One-Half. The Bessel function $J_{l+1/2}(z)$, $l > 0$, is given by [3, p. 298],

$$J_{l+1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} [R_{l,1/2}(z)\sin z - R_{l-1,3/2}(z)\cos z], \tag{4.1}$$

where [3, p. 296],

$$R_{l,\nu}(z) = \sum_{n=0}^{[l/2]} (-1)^n \binom{l - n}{n} \frac{\Gamma(\nu + l - n)}{\Gamma(\nu + n)} (\tfrac{1}{2}z)^{-l+2n}. \tag{4.2}$$

Also, [3, p. 40],

$$J_{l+1/2}(z) = (\tfrac{1}{2}z)^{l+1/2} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!\, \Gamma(l + k + 3/2)} = z^{l+1/2} P(z), \quad \text{where } P(0) \neq 0. \tag{4.3}$$

By (4.1), (4.3) we get

$$(4.4)\qquad (\pi/2)^{1/2}z^{2l+1}P(z) = [z^l R_{l,1/2}(z)]\sin z - [z^{l-1}R_{l-1,3/2}(z)]\, z\cos z.$$

Now, $z^l R_{l,1/2}(z) = 2^l g_{l,-1/2}(\frac{1}{4}z^2)$, and $z^{l-1}R_{l-1,3/2}(z) = 2^{l-1}g_{l-1,1/2}(\frac{1}{4}z^2)$, where

$$(4.5)\qquad g_{l,\nu}(\zeta) = \sum_{k=0}^{[l/2]} \binom{l-k}{k} \frac{\Gamma(\nu+l-k+1)}{\Gamma(\nu+k+1)}\zeta^k$$

are the modified Lommel polynomials [3, p. 303].

Since

$$2^l g_{l,-1/2}(0) = \frac{2^l\Gamma(l+\frac{1}{2})}{\Gamma(\frac{1}{2})} \quad\text{and}\quad 2^{l-1}g_{l-1,1/2}(0) = \frac{2^{l-1}\Gamma(l+\frac{1}{2})}{\Gamma(\frac{3}{2})} = 2^l g_{l,-1/2}(0),$$

we get that

$$(4.6)\qquad \begin{aligned} f_l(z) &= \frac{\Gamma(\frac{1}{2})}{2^l\Gamma(l+\frac{1}{2})}\left(\frac{\pi}{2}\right)^{1/2} z^{2l+1}P(z) \\ &= \frac{\Gamma(\frac{1}{2})}{2^l\Gamma(l+\frac{1}{2})}\left(\frac{\pi}{2}\right)^{1/2} z^{l+1/2}J_{l+1/2}(z) \\ &= B_m(z^2)\sin z - zA_n(z^2)\cos z, \end{aligned}$$

is of the form (3.4), with $r = 1$, $b_0 = a_0 = 1$,

$$(4.7)\qquad B_m(z^2) = \frac{\Gamma(\frac{1}{2})}{\Gamma(l+\frac{1}{2})}\, g_{l,-1/2}(\tfrac{1}{4}z^2)$$

and

$$(4.8)\qquad A_n(z^2) = \frac{\Gamma(\frac{1}{2})}{\Gamma(l+\frac{1}{2})}\, g_{l-1,1/2}(\tfrac{1}{4}z^2),$$

where

$$(4.9)\qquad n = m = (l-1)/2,\quad l \text{ odd},\qquad n = l/2,\quad n = m-1,\quad l \text{ even}.$$

We can therefore apply the results of Section 3 to $J_{l+1/2}(z)$, $l > 0$. It is well known [3, p. 482] that the Bessel function $J_\nu(z)$ has only real zeros for $\nu > -1$, and this is therefore true for $J_{l+1/2}(z)$, $l > 0$. The following theorem is therefore of interest.

THEOREM 3. *If in $J_{l+1/2}(z)$, for $l > 2$, we perturb any one of the coefficients in $R_{l,1/2}(z)$ or $R_{l-1,3/2}(z)$, the resultant function has nonreal zeros.*

To prove this theorem, we shall first prove two lemmas which are of interest themselves.

LEMMA 3. *The coefficients $b_0, b_1, \cdots, b_m, a_0, a_1, \cdots, a_n$, of* (4.7), (4.8) *are uniquely determined by the following set of linear nonhomogenous equations*

$$(4.10)\qquad b_0 = 1,\qquad c_j = d_j,\qquad j = 0, 1, \cdots, l-1,$$

where c_j, d_j are defined as in (3.8).

Proof. $b_0 = 1$ is a condition of (3.4), which is satisfied by $f_l(z)$ in (4.6). From (3.10) we have

$$(4.11)\qquad f_l(z) = C_0 z^q \prod_{k=1}^{\infty}(1 - z^2/\alpha_k^2) = z^q P_1(z),$$

where $P_1(0) \neq 0$, and $q = 2i + 1$ if $c_j - d_j = 0$ for $j = 0, 1, \cdots, i - 1$, and $c_i - d_i \neq 0$. From (4.6), $f_l(z) = z^{2l+1}P_2(z)$, with $P_2(0) \neq 0$. It follows that $q = 2l + 1$, and therefore $i = l$, and (4.10) is satisfied. The number of equations in (4.10) is $l + 1$, and the number of unknowns (counting b_0) is $n + m + 2$ which by (4.9) is equal to $l + 1$, so that (4.10) is a set of $l + 1$ nonhomogenous linear equations in $l + 1$ unknowns. For uniqueness, see Appendix.

LEMMA 4. *Let $\bar{b}_0\,(=1), \bar{b}_1, \cdots, \bar{b}_m, \bar{a}_0, \bar{a}_1, \cdots, \bar{a}_n$, be the (unique) solution of* (4.10). *Let $b_s = \bar{b}_s + \epsilon$, $a_s = \bar{a}_s + \delta$, $0 \leqq s \leqq m$, and $b_j = \bar{b}_j$, $a_j = \bar{a}_j$ for $j \neq s$.* [*If $n = m - 1$, see* (4.9), *then $\bar{a}_m = \delta = 0$.*] *Let*

$$(4.12)\qquad f_l(z, s, \epsilon, \delta) = \frac{1}{b_0}\left[\sum_{k=0}^{m} b_k z^{2k} \sin z - z \sum_{k=0}^{n} a_k z^{2k} \cos z\right],$$

and let $S_k^l(s, \epsilon, \delta)$ denote the sums of the nonzero zeros (1.1) *of $f_l(z, s, \epsilon, \delta)$. Then*

$$(4.13)\qquad S_j^l(s, \epsilon, 0) = \pi^{-2j-2}\zeta(2j + 2), \qquad j \leqq l - s - 2,$$

and

$$(4.14)\qquad S_j^l(s, 0, \delta) = \pi^{-2j-2}(2^{2j+2} - 1)\zeta(2j + 2), \qquad j \leqq l - s - 2.$$

In particular, when $j = 0$, (4.13) *holds for $l > 2$, and $l = 2$, $s = 0$, and* (4.14) *holds for $l \geqq 2$.*

Proof. $f_l(z, s, \epsilon, \delta)$ is of the form (3.4), with $r = 1$. It follows from (3.8) and (4.10), that $c_j - d_j = 0$ for $j < s$, and

$$(4.15)\qquad \begin{aligned} c_s - d_s &= (-1)^s\left[\sum_{k=0}^{s} \frac{(-1)^k \bar{b}_k}{(2s - 2k + 1)!} + (-1)^s\epsilon - \sum_{k=0}^{s} \frac{(-1)^k \bar{a}_k}{(2s - 2k)!} - (-1)^s\delta\right] \\ &= \epsilon - \delta \qquad (=\epsilon, \text{ if } s = m = n + 1). \end{aligned}$$

For $\epsilon \neq \delta$, we can write, by (3.10),

$$(4.16)\qquad f_l(z, s, \epsilon, \delta) = (\epsilon - \delta)z^{2s+1} \prod_{k=1}^{\infty} (1 - z^2/\alpha_k^2),$$

and by (3.22) we have

$$(4.17)\qquad \sum_{k=0}^{j} S_k^l(s, \epsilon, \delta)[c_{s+j-k} - d_{s+j-k}] = -(j + 1)[c_{s+j+1} - d_{s+j+1}], \qquad j = 0, 1, 2, \cdots.$$

If $s + t \leqq l - 1$, then by (4.10) and (3.8),

$$(4.18)\qquad c_{s+t} - d_{s+t} = \frac{(-1)^t}{(2t + 1)!}\cdot\epsilon - \frac{(-1)^t}{(2t)!}\cdot\delta,$$

and so for $s + j + 1 \leqq l - 1$, (4.17) becomes

$$(4.19)\qquad \sum_{k=0}^{j} S_k^l(s, \epsilon, \delta)(-1)^k\left[\frac{\epsilon - (2j - 2k + 1)\delta}{(2j - 2k + 1)!}\right] = (j + 1)\frac{\epsilon - (2j + 3)\delta}{(2j + 3)!},$$

which is equation [1, (21)] with the parameter $k = \delta/\epsilon \neq 1$. In particular, if $\epsilon \neq 0$, $\delta = 0$, this is [1, (21)] with $k = 0$, and so by [1, (24)], we have (4.13). If $\delta \neq 0$, $\epsilon = 0$, this is [1, (21)] with $k \to \infty$, and so by [1, (28)] we have (4.14). Since $s \leqq m$, it follows from (4.9) that when $j = 0$, (4.13) holds for $l > 2$ and $l = 2$, $s = 0$, and since $s \leqq n$ in (4.14), (4.14) holds for $l \geqq 2$, if $j = 0$.

Proof of Theorem 3. The nonzero zeros of $J_{l+1/2}(z)$ are the same as those of $f_l(z)$ in (4.6), and perturbing a coefficient of $R_{l,1/2}(z)$ or $R_{l-1,3/2}(z)$ is equivalent to perturbing a coefficient of $B_m(z^2)$ or $A_n(z^2)$ in (4.6). We can therefore look at the function $f_l(z)$. Let b_j $(j = 1, 2, \cdots, m)$, a_j $(j = 1, 2, \cdots, n)$ be defined as in Lemma 4. Since $l > 2$, we have by (4.13)

$$S_0^l(s, \epsilon, 0) = \pi^{-2}\zeta(2) = \tfrac{1}{6}, \qquad \epsilon \neq 0, \tag{4.20}$$

and by (4.14),

$$S_0^l(s, 0, \delta) = \pi^{-2}3\zeta(2) = \tfrac{1}{2}, \qquad \delta \neq 0. \tag{4.21}$$

From (4.20) it follows that

$$\lim_{\epsilon \to 0} S_0^l(s, \epsilon, 0) = \tfrac{1}{6}. \tag{4.22}$$

The roots $\pm\alpha_1, \pm\alpha_2, \cdots$ depend continuously on ϵ. If $\epsilon \neq 0$ $(\delta = 0)$, we have from (4.16) that the multiplicity at zero is of order $2s + 1$, but when $\epsilon = 0$ $(\delta = 0)$, we have from (4.6) that the multiplicity at zero is $2l + 1$. Since the roots occur in pairs $\pm\alpha$, we must have

$$\lim_{\epsilon \to 0} \alpha_j^2 = 0, \qquad j = 1, 2, \cdots, l - s. \tag{4.23}$$

But

$$\tfrac{1}{6} = S_0^l(s, \epsilon, 0) = \sum_{k=1}^{l-s} \alpha_k^{-2} + \sum_{k=l-s+1}^{\infty} \alpha_k^{-2},$$

and

$$\lim_{\epsilon \to 0} \sum_{k=l-s+1}^{\infty} \alpha_k^{-2} = \frac{1}{2(2l + 3)},$$

see [3, p. 502] or [4]. We therefore get that $Q(\epsilon) = \sum_{k=1}^{l-s} \alpha_k^{-2}$ stays finite as $\epsilon \to 0$; If all α_k^2 in $Q(\epsilon)$ tended to zero through positive values, then $\lim_{\epsilon \to 0} Q(\epsilon) = \infty$. So that, when $\epsilon \neq 0$ $(|\epsilon| \ll 1)$, we must have at least one root, α_j, for which α_j^2 is not positive, i.e., α_j is nonreal. The same argument applied to $S_0^l(s, 0, \delta)$ by (4.21) again yields that if $\delta \neq 0$ $(|\delta| \ll 1)$ we must have nonreal roots, completing the proof.

Remark. From the proof and Lemma 4, it is obvious that Theorem 3 holds also when $l = 2$, $s = 0$. (For $l = 2$, we only have a_0, b_0 and b_1, and the theorem does not apply to b_1.)

There are three cases which Theorem 3 does not cover, if $l + \frac{1}{2} > 0$; the cases $l = 0$, $l = 1$, and $l = 2$, $s = 1$ (only b_1). The case $l = 0$ is not of the form we are discussing. Indeed $(\pi z/2)^{1/2} J_{1/2}(z) = \sin z$, so we only have $b_0 = 1$, and obviously the roots do not depend on changing b_0 to any nonzero constant. The other two cases are indeed different from the cases covered by Theorem 3, and we have the following.

THEOREM 4. *Under the same definitions as in Lemma* 4,

(1) $f_1(z, 0, -\epsilon, 0)$ *and* $f_1(z, 0, 0, \delta)$ *have only real zeros for* $\epsilon \geqq 0$, $\delta \geqq 0$, *and a pair of imaginary roots for* $\epsilon < 0$, $\delta < 0$ $(|\epsilon| \ll 1, |\delta| \ll 1)$.

(2) $f_2(z, 1, -\epsilon, 0)$ *has only real roots for* $\epsilon \geqq 0$, *and a pair of imaginary roots for* $\epsilon < 0$ ($|\epsilon| \ll 1$).

Proof. *Part* (1). This is the problem of [1, Section 1], with the parameter $k = a_0/b_0$ there, and (1) follows.

Part (2). Note that

$$f_2(z) = (1 - \tfrac{1}{3}z^2)\sin z - z\cos z. \tag{4.24}$$

If we look at the function $g(z) = (k_1 - k_2 z^2)\sin z - z\cos z$, the roots, $\pm\alpha$ of $g(z)$ arise from the Sturm-Liouville system

$$\begin{aligned} u'' + \alpha^2 u &= 0, \\ Cu(0) + Du'(0) &= 0, \\ Eu(1) - Fu'(1) &= 0, \quad ED + CF \neq 0, \end{aligned} \tag{4.25}$$

where $k_1 = CE/(ED + CF)$, $k_2 = FD/(ED + CF)$.

By Sturm-Liouville theory, α^2 are all real, and therefore $f_2(z, 1, -\epsilon, 0)$ may have either real or imaginary roots. An imaginary root $z = \pm ix$, $x > 0$, of $f_2(z, 1, -\epsilon, 0)$ should satisfy the equation

$$y_1(x) = y_2(x), \tag{4.26}$$

where $y_1(x) = \tanh x$, $y_2(x) = x/(1 + cx^2)$, $c = \frac{1}{3} + \epsilon$. Since $l - s = 2 - 1 = 1$ only one such $x > 0$ can exist (for ϵ small enough). $y_1(0) = y_2(0) = 0$ and $\lim_{x\to\infty} y_1(x) = 1$, $\lim_{x\to\infty} y_2(x) = 0$. As $y_1(x)$, $y_2(x)$ may intersect only once, for $x > 0$, it is necessary and sufficient that $y_2(x) > y_1(x)$ in some interval $(0, \eta^2)$, for the existence of a positive solution of (4.26). Checking derivatives at zero, one finds $y_1^{(j)}(0) = y_2^{(j)}(0)$ for $j = 0, 1, 2$, and $y_1^{(3)}(0) = -2$, $y_2^{(3)}(0) = -6c$, and therefore, there exists a positive root of (4.26) iff $-6c > -2$, i.e., $c < \frac{1}{3}$. This concludes the proof of Part (2).

To get an idea of the asymptotic behavior of the roots which split away from zero when ϵ (or δ) tend to zero, note that by the definitions, $\lim_{\epsilon\to 0} f_l(z, s, \epsilon, 0) = \lim_{\delta\to 0} f_l(z, s, 0, \delta) = f_l(z)$. From (4.16),

$$\begin{aligned} f_l(z, s, \epsilon, 0) &= \epsilon z^{2s+1} \prod_{k=1}^{\infty} (1 - z^2/\alpha_k^2) \\ &= z^{2l+1} \prod_{k=l-s+1}^{\infty} (1 - z^2/\alpha_k^2) \prod_{k=1}^{l-s} (1/z^2 - 1/\alpha_k^2)\epsilon \\ &= z^{2l+1} \prod_{k=l-s+1}^{\infty} (1 - z^2/\alpha_k^2) \prod_{k=1}^{l-s} (1 - \alpha_k^2/z^2)(-1)^{l-s}\, \epsilon \prod_{k=1}^{l-s} \alpha_k^{-2}. \end{aligned} \tag{4.27}$$

Take the limit $\epsilon \to 0$ of (4.27) and use (4.23), (4.6), the continuous dependence of the roots on ϵ, and (3.10), to get

$$f_l(z) = f_l(z) \lim_{\epsilon\to 0} (-1)^{l-s}\epsilon(c_l - d_l)^{-1} \prod_{k=1}^{l-s} \alpha_k^{-2},$$

or

$$\lim_{\epsilon\to 0} \prod_{k=1}^{l-s} \alpha_k^{-2}\cdot\epsilon = (-1)^{l-s}(c_l - d_l).$$

From (4.3), (4.6) we get that

$$c_l - d_l = (2l+1)^{-1}[2^l l!/(2l)!]^2, \tag{4.28}$$

and we get

$$\lim_{\epsilon\to 0} \prod_{k=1}^{l-s} \alpha_k^{-2}\cdot\epsilon = (-1)^{l-s}(2l+1)^{-1}[2^l l!/(2l)!]^2. \tag{4.29}$$

Again, when $\epsilon = 0$, $\delta \neq 0$, (4.29) holds with ϵ replaced by $(-\delta)$.

Example. $l = 3$, $s = 1$, $\epsilon < 0$. By Theorem 3, we know we must have at least one nonreal root. By (4.29), $\lim_{\epsilon\to 0}\epsilon\alpha_1^{-2}\alpha_2^{-2} = 1/1575$, and from Theorem 3, we have $\lim_{\epsilon\to 0}\alpha_1^{-2} + \alpha_2^{-2} = \frac{1}{9}$. If $|\epsilon| \ll 1$, we must therefore have $\alpha^2 = \alpha_1^2 > 0$, and $-\beta^2 = \alpha_2^2 < 0$, and so the two double roots at zero separate to $\pm\alpha$ ($\alpha > 0$) along the real axis, and $\pm i\beta$ ($\beta > 0$) along the imaginary axis. For $|\epsilon| \ll 1$, $\alpha^2\beta^2 \approx 1575(-\epsilon)$, $\beta^2 - \alpha^2 \approx 175(-\epsilon)$, or $\alpha^2 \approx \beta^2 \approx 40\,|\epsilon|^{1/2} + O(\epsilon)$.

We have [3, p. 298],

$$(-1)^l J_{-l-1/2}(z) = (2/\pi z)^{1/2}[R_{l-1,3/2}(z)\sin z + R_{l,1/2}(z)\cos z], \tag{4.30}$$

and so

$$\frac{\Gamma(\frac{1}{2})}{2^l\Gamma(l+\frac{1}{2})}(-1)^l\left(\frac{\pi}{2}\right)^{1/2} z^{l+1/2} J_{-l-1/2}(z), \qquad l > 0,$$

is of the form (3.5) with $r = 1$, $m = n = (l-1)/2$ for l odd and $n = l/2$, $m = n-1$ for l even, and the coefficients are given by

$$b_j = \bar{a}_j, \qquad a_j = -\bar{b}_j, \qquad j = 1, 2, \cdots, n, \tag{4.31}$$

where $\bar{a}_j$, $\bar{b}_j$ ($j = 1, \cdots, n$) are the solutions of (4.10).

Since $r = 1$, (3.26) holds with $t = 0$, and from (3.27), (4.31), we get

$$S_0 = -\tfrac{1}{2} - \bar{b}_1, \quad S_1 = \tfrac{1}{2} + \bar{b}_1^2 + 2\bar{b}_2 + 2\bar{b}_1 - 2\bar{a}_1. \tag{4.32}$$

Appendix. We want to prove that there exists a unique solution $b_0, \cdots, b_m$, $a_0, \cdots, a_n$ to the set of equations

$$b_0 = 1,$$

$$c_s - d_s = (-1)^s\left[\sum_{k=0}^{m}\frac{(-1)^k b_k}{(2s-2k+1)!} - \sum_{k=0}^{n}\frac{(-1)^k a_k}{(2s-2k)!}\right] = 0, \tag{A.1}$$

$$s = 0, 1, 2, \cdots, l-1,$$

$$m = n = (l-1)/2, \quad \text{for } l \text{ odd}, \quad \text{and} \quad m = l/2, \quad n = m-1 \quad \text{for } l \text{ even}.$$

Note that we can replace n by m, and add the equation $a_m = 0$, for l even. We know that for any $l \geqq 0$ there exists a solution to (A.1) given by the coefficients in (4.7), (4.8). From (3.9), (A.1) holds iff

$$f(z) = \sin z\sum_{k=0}^{m} b_k z^{2k} - z\cos z\sum_{k=0}^{n} a_k z^{2k}, \qquad (b_0 = 1), \tag{A.2}$$

has a zero of multiplicity at least $2l + 1$ at zero, and actually, for the solution we know it is exactly of multiplicity $2l + 1$, by (4.28). Thus, (A.1) admits a unique solution iff any other choice of the coefficients, b_0 $(=1)$, $b_1, \cdots, b_m, a_0, a_1, \cdots, a_n$, would have the corresponding function $f(z)$ in (A.2), with a zero of multiplicity less than $2l + 1$, at zero. We therefore rephrase what we want to prove as follows:

THEOREM 5. *There exists a unique set of coefficients* $b_0, \cdots, b_m, a_0, \cdots, a_m$ *such that, at zero,* $f(z)$, *defined by* (A.2), *has a zero of multiplicity* $4m + 1$ $(l = 2m)$ *if we demand* $a_m = 0$, *and multiplicity* $4m + 3$ $(l = 2m + 1)$ *otherwise, and such that* $b_0 = 1$.

Proof. By induction.

(1) $m = 0$. From (A.1), $b_0 = 1$ for $l = 0$, and $a_0 = b_0 = 1$ for $l = 1$, and the solution is unique, so the theorem holds.

(2) The induction hypothesis is that the theorem is true for $m = 0, 1, \cdots, i$. We want to prove it for $m = i + 1$. Note that the first two equations in (A.1) are always $b_0 = 1$, $a_0 = b_0$, and so

$$b_0 = a_0 = 1 \quad \text{for all } m > 0. \tag{A.3}$$

Part a. Let $\bar{b}_0, \cdots, \bar{b}_{i+1}, \bar{a}_0, \cdots, \bar{a}_i, \bar{a}_{i+1}$ $[=0]$, and $b_0, \cdots, b_{i+1}, a_0, \cdots, a_{i+1}$ $[=0]$ be two distinct solutions of (A.1) for $l = 2i + 2$. It follows from (A.2) and (3.9) that we can write

$$f_1(z) = \sin z \sum_{k=0}^{i+1} \bar{b}_k z^{2k} - z \cos z \sum_{k=0}^{i} \bar{a}_k z^{2k} = z^{4i+5} R_1(z^2) \tag{A.4}$$

and

$$f_2(z) = \sin z \sum_{k=0}^{i+1} b_k z^{2k} - z \cos z \sum_{k=0}^{i} a_k z^{2k} = z^{4i+5} R_2(z^2). \tag{A.5}$$

Subtract (A.5) from (A.4) to get

$$\sin z \sum_{k=0}^{i+1} (\bar{b}_k - b_k) z^{2k} - z \cos z \sum_{k=0}^{i} (\bar{a}_k - a_k) z^{2k} = z^{4i+5}[R_1(z^2) - R_2(z^2)], \tag{A.6}$$

and, by (A.3), we must have

$$a_0 = \bar{a}_0 = b_0 = \bar{b}_0 = 1. \tag{A.7}$$

From (A.1) it follows that $\bar{b}_k - b_k = 0$ for all $k \leqq j - 1 \leqq m + 1$ iff $\bar{a}_k - a_k = 0$ for all $k \leqq j - 1 \leqq m + 1$. Suppose $\bar{b}_k - b_k = 0$ for $k = 0, 1, 2, \cdots, j - 1$, and $\bar{b}_j - b_j \neq 0$. $j \geqq 1$ because of (A.7), and $j \leqq i$ since the two solutions are distinct, and $\bar{a}_{i+1} = a_{i+1} = 0$. Divide (A.6) by z^{2j} to get

$$\begin{aligned}\sin z \sum_{k=0}^{i+1-j} [\bar{b}_{k+j} - b_{k+j}] z^{2k} - z \cos z \sum_{k=0}^{i-j} [\bar{a}_{k+j} - a_{k+j}] z^{2k} \\ = z^{4i+5-2j}[R_1(z^2) - R_2(z^2)].\end{aligned} \tag{A.8}$$

Divide through by $\bar{b}_j - b_j$ $(\neq 0)$, and the left-hand side of (A.8) is of the form (A.2) with $0 \leqq m \leqq i + 1 - j \leqq i$, and $a_m = 0$. From the induction hypothesis, it follows that the left-hand side of (A.8) can have a zero of multiplicity at most $4m + 1 \leqq 4i + 5 - 4j$ at zero. But the right-hand side of (A.8) has multiplicity of at least $4i + 5 - 2j > 4i + 5 - 4j$, a contradiction. Thus, (A.1) admits a unique solution for $l = 2i + 2$.

Part b. Let $\bar{b}_0, \cdots, \bar{b}_{i+1}, \bar{a}_0, \cdots, \bar{a}_{i+1}$ and $b_0, \cdots, b_{i+1}, a_0, \cdots, a_{i+1}$ be two distinct solutions of (A.1) for $l = 2i + 3$. Repeat the same process as in Part a to get the equation

$$\text{(A.9)} \quad \begin{aligned} \sin z \sum_{k=0}^{i+1-j} [\bar{b}_{k+j} - b_{k+j}]z^{2k} - \sum_{k=0}^{i+1-j} [\bar{a}_{k+j} - a_{k+j}]z^{2k} \\ = z^{4i+7-2j}[R_3(z^2) - R_4(z^2)], \end{aligned}$$

and $1 \leqq j \leqq m + 1$. Again divide by $\bar{b}_j - b_j$ ($\neq 0$), and get the left-hand side of (A.9) in the form (A.2) with $0 \leqq m \leqq i + 1 - j \leqq i$. Since we do not require $a_m = 0$, we have by the induction hypothesis a zero of multiplicity at most $4m + 3 \leqq 4i + 7 - 4j$ for the left-hand side of (A.9) at zero. The right-hand side has, at zero, a zero of multiplicity at least $4i + 7 - 2j > 4i + 7 - 4j$, since $j \geqq 1$, a contradiction. This concludes the proof.

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1. N. Liron, "Some infinite sums," *SIAM J. Math. Anal.*, v. 2, 1971, pp. 105–112.
2. R. P. Boas, *Entire Functions*, Academic Press, New York, 1954. MR **16**, 914.
3. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, Cambridge; Macmillan, New York, 1944. MR **6**, 64.
4. N. Kishore, "The Rayleigh function," *Proc. Amer. Math. Soc.*, v. 14, 1963, pp. 527–533. MR **27** #1633.