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Series Expansions of $W_{k,m}(z)$ Involving Parabolic Cylinder Functions

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Abstract. In this paper, an explicit error bound is obtained for an expansion of the Whittaker function, $W_{k,m}(z)$, in series of parabolic cylinder functions. It is also shown that the Whittaker function may be asymptotically represented as the sum of two products where one product involves a parabolic cylinder function and the other product involves the first-order derivative of this function.

1. Introduction. It has been known from the work of Erdélyi [2, p. 124] that the Whittaker function $W_{k,m}(z)$ can be expanded in terms of the parabolic cylinder functions, $D_\nu(z)$. The usual form of this expansion is given by

$$W_{k,m}(z^2) = 2^{1/4-k}\sqrt{z}\left\{\sum_{\lambda=0}^{p-1}\frac{(2m,\lambda)}{(2z\sqrt{2})^\lambda}D_{2k-\lambda-1/2}(z\sqrt{2}) + R_p\right\}, \tag{1.1}$$

where R_p denotes the remainder term. Unless $2m$ is half of an odd integer, the series is divergent. In the case when $2m$ is half of an odd integer the series terminates at a finite stage.

It is therefore desirable to have an accurate estimate of the remainder R_p. In [3], Erdélyi obtained an explicit upper bound for R_p in the case when $|\arg z| < \frac{1}{4}\pi$ and p was large. However, a detailed study of the behavior of this bound and the behavior of $D_\nu(z\sqrt{2})$ indicates that Erdélyi's estimate is insufficient to yield the asymptotic nature of the series.

The purpose of this paper is two-fold. We first show in Section 2 that Erdélyi's result can be considerably improved (see Theorem 1), and that the new estimate is sufficient to indicate the asymptotic character of the series in (1.1). Next, we obtain, in Section 3, an asymptotic expansion of the form

$$W_{k,m}(z^2) \sim D_{2k-1/2}(z\sqrt{2})\sum_{s=0}^{\infty}\frac{c_s}{z^{2s-1/2}} + D'_{2k-1/2}(z\sqrt{2})\sum_{s=0}^{\infty}\frac{d_s}{z^{2s+1/2}}, \tag{1.2}$$

where the coefficients c_s and d_s are constants independent of z. It should be pointed out that Erdélyi [3] did mention the possibility of such a representation in the special case when $2m$ is half of an odd integer. However, the result for the general case appears to be new and its proof requires our estimate for R_p given in (2.7).

2. Estimation of R_p. The Whittaker function $W_{k,m}$ has the integral representation [2, p. 73],

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$$W_{k,m}(z^2) = z \exp\,[z^2/2 + (m + \tfrac{1}{2} - k)\pi i] \int_\Gamma \exp\,[-u^2] H_{2m}^{(1)}(2zu) u^{2k}\, du, \tag{2.1}$$

where $H_\nu^{(1)}(\omega)$ is the Hankel function of order ν, and the contour Γ runs from $-\infty$ to $+\infty$ and passes above the singularity at the origin. If $-\frac{1}{2}\pi < \arg \omega < \frac{3}{2}\pi$, it is well known [4, p. 219] that

$$H_\nu^{(1)}(\omega) = \left(\frac{2}{\omega\pi}\right)^{1/2} \exp\,[i(\omega - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)]\left[\sum_{\lambda=0}^{p-1} \frac{(-1)^\lambda(\nu, \lambda)}{(2i\omega)^\lambda} + r_p(\omega)\right], \tag{2.2}$$

provided that $p + \frac{1}{2} > |\mathrm{Re}\, \nu|$. The remainder r_p satisfies the inequality

$$|r_p(\omega)| \leqq \theta(\arg \omega) \left|\frac{\cos \nu\pi}{\cos \mathrm{Re}\, \nu\pi}\right| \frac{|(\mathrm{Re}\, \nu, p)|}{(2\,|\omega|)^p} \tag{2.3}$$

where

$$\begin{aligned} \theta(\varphi) &= \sec \varphi && \text{if } -\pi/2 < \varphi \leqq 0, \\ &= 1 && \text{if } 0 \leqq \varphi \leqq \pi, \\ &= -\sec \varphi && \text{if } \pi \leqq \varphi < \tfrac{3}{2}\pi. \end{aligned} \tag{2.4}$$

Substituting for $H_\nu^{(1)}$ in (2.1) and integrating term-by-term, we obtain, as did Erdélyi, the series representation (1.1) and the expression

$$R_p = \frac{1}{\sqrt{\pi}}\, 2^{k-1/4} \exp\,[(\tfrac{1}{4} - k)\pi i + \tfrac{1}{2}z^2]\bar{R}_p, \tag{2.5}$$

where

$$\bar{R}_p = \int_\Gamma \exp\,[-u^2 + 2izu] u^{2k-1/2} r_p(2zu)\, du. \tag{2.6}$$

THEOREM 1. *If* $p + \frac{1}{2} > 2\,|\mathrm{Re}\, m|$ *and* $p + \frac{1}{2} \geqq 2\,\mathrm{Re}\, k$, *then*

$$|R_p| \leqq B_p \sec^{-3/2}(\arg z^2)\, |\exp\,[-z^2/2] z^{2k-2p-1/2}| \tag{2.7}$$

for $|\arg z| < \frac{1}{4}\pi$, *where*

$$B_p = 2^{\mathrm{Re}\, k - 1/4} \exp\,[|\mathrm{Im}\, k|\, \pi] \frac{|\cos 2m\pi|}{|\cos 2\,\mathrm{Re}\, m\pi|} \cdot \frac{|(2\,\mathrm{Re}\, m, p)|}{4^p}. \tag{2.8}$$

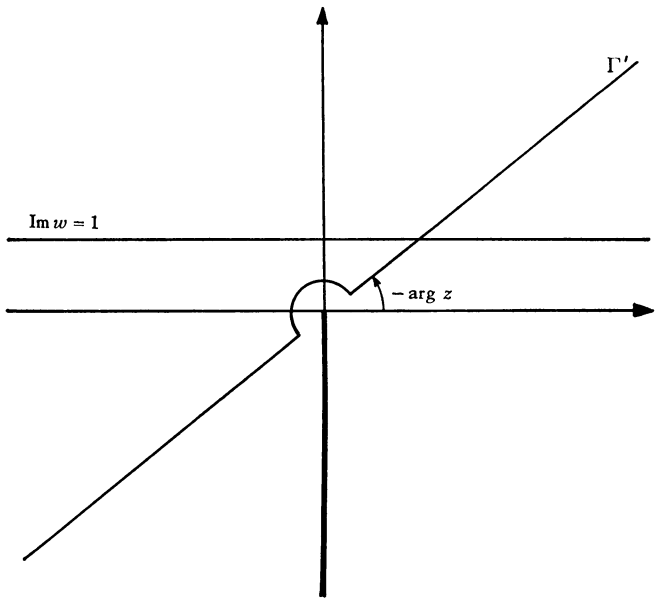
Proof. On making the substitution $u = zw$, we can rewrite $\bar{R}_p$ as

$$\bar{R}_p = z^{2k+1/2} \int_{\Gamma'} \exp\,[-z^2(w^2 - 2iw)] w^{2k-1/2} r_p(2wz^2)\, dw, \tag{2.9}$$

where Γ' is the image of Γ. From (2.2), it is readily seen that the integrand in (2.9) is regular in the w-plane cut along the negative imaginary axis. Then, according to Cauchy's integral theorem, the integral along the path Γ' can be replaced by an integral along the straight line $\mathrm{Im}\, w = 1$. Thus, formula (2.9) becomes

$$\bar{R}_p = z^{2k+1/2} \int_{-\infty}^{\infty} \exp\,[-z^2(x^2 + 1)](x + i)^{2k-1/2} r_p(2z^2(x + i))\, dx. \tag{2.10}$$

The cut w-plane and the paths of integration are illustrated below.



w-plane

To estimate $\bar{R}_p$, we use (2.3). Hence,

$$|\bar{R}_p| \leqq \left|\frac{\cos 2m\pi}{\cos 2 \operatorname{Re} m\pi}\right| \cdot \frac{|(2 \operatorname{Re} m, p)|}{4^p} \tag{2.11}$$
$$\cdot \sec(\arg z^2)\, |\exp[-z^2] z^{2k-2p+1/2}| \int_{-\infty}^{\infty} |\exp[-z^2x^2](x+i)^{2k-p-1/2}\, dx|.$$

Since $p + \frac{1}{2} \geqq 2 \operatorname{Re} k$, the integral on the left-hand side of (2.11) is dominated by

$$\beta_k \sqrt{(\pi/\operatorname{Re} z^2)}, \tag{2.12}$$

where β_k is 1 or $\exp\{-\pi(2 \operatorname{Im} k)\}$ according to $\operatorname{Im} k \geqq 0$ or $\operatorname{Im} k \leqq 0$. Therefore

$$|\bar{R}_p| \leqq \beta_k \sqrt{\pi} \left|\frac{\cos 2m\pi}{\cos 2 \operatorname{Re} m\pi}\right| \cdot \frac{|(2 \operatorname{Re} m, p)|}{4^p} \tag{2.13}$$
$$\cdot \sec^{3/2}(\arg z^2)\, |\exp[-z^2] \cdot z^{2k-2p-1/2}|.$$

A combination of the results (2.5) and (2.13) then gives the desired estimate (2.7). This completes the proof of Theorem 1.

Remark. When $p + \frac{1}{2} > 2\,|\operatorname{Re} m|$ and $p + \frac{1}{2} < 2 \operatorname{Re} k$, one can obtain an upper bound which is similar to (2.7). In this case, the integral in (2.11) is dominated by

$$2\beta_k \int_0^{\infty} \exp[-(\operatorname{Re} z^2)x^2](x^2+1)^{2 \operatorname{Re} k-p-1/2}\, dx. \tag{2.14}$$

If $\operatorname{Re} z^2 \geqq 1$, then the last integral in its turn is dominated by

$$\frac{2\beta_k}{\sqrt{(\operatorname{Re} z^2)}} \int_0^{\infty} \exp[-t^2](t^2+1)^{2 \operatorname{Re} k-p-1/2}\, dt = \frac{\beta_k'}{\sqrt{(\operatorname{Re} z^2)}}. \tag{2.15}$$

Therefore, a constant B_p can again be found such that the result (2.7) holds for $\operatorname{Re} z^2 \geqq 1$.

3. Proof of (1.2). If in Theorem 1 we restrict z to the sector $|\arg z| \leqq \frac{1}{4}\pi - \delta$, $\delta > 0$, then (2.7) gives

$$R_p = O(\exp[-z^2/2]z^{2k-2p-1/2}) \quad \text{as } |z| \to \infty, \tag{3.1}$$

provided that $p > 2\,|\operatorname{Re} m| - \frac{1}{2}$. Furthermore, (1.1) can now be written in the form

$$\begin{aligned} &W_{k,m}(z^2) \\ &\quad = 2^{1/4-k}\sqrt{z}\left\{\sum_{\lambda=0}^{p-1} \frac{(2m,\lambda)}{(2z\sqrt{2})^\lambda} D_{2k-\lambda-1/2}(z\sqrt{2}) + O(\exp[-z^2/2]z^{2k-2p-1/2})\right\}, \end{aligned} \tag{3.2}$$

uniformly in $\arg z$, as $z \to \infty$ in $|\arg z| \leqq \frac{1}{4}\pi - \delta$. The constant implied in symbol 0 is independent of $\arg z$. Since

$$D_\nu(z\sqrt{2}) \sim (z\sqrt{2})^\nu \exp[-z^2/2] \tag{3.3}$$

as $z \to \infty$ in $|\arg z| < \frac{3}{4}\pi$, the condition $p > 2\,|\operatorname{Re} m| - \frac{1}{2}$ in Theorem 1 can also be removed. Hence, (3.2) holds for all $p \geqq 1$.

THEOREM 2. *As $z \to \infty$ in $|\arg z| \leqq \frac{1}{4}\pi - \delta$, then, for any fixed $N \geqq 0$,*

$$\begin{aligned} 2^{k-1/4}z^{-1/2}W_{k,m}(z^2) &= D_{2k-1/2}(z\sqrt{2})\left\{\sum_{l=0}^{N} \frac{a_l}{z^{2l}} + O\left(\frac{1}{z^{2N+2}}\right)\right\} \\ &\quad + \frac{D'_{2k-1/2}(z\sqrt{2})}{z}\left\{\sum_{l=0}^{N} \frac{b_l}{z^{2l}} + O\left(\frac{1}{z^{2N+2}}\right)\right\} \end{aligned} \tag{3.4}$$

uniformly with respect to $\arg z$. *The coefficients a_l and b_l are given in* (3.12).

To prove this theorem we shall use the following result given in [5].

LEMMA. *For each $\lambda \geqq 0$, we have*

$$D_{\nu-\lambda}(z) = D_\nu(z)P_\lambda(z) + D'_\nu(z)Q_{\lambda-1}(z), \tag{3.5}$$

where $P_\lambda(z)$ and $Q_{\lambda-1}(z)$ are polynomials of the form

$$P_\lambda(z) = \sum_{k=0}^{\leq\lambda/2} p_{\lambda,k}z^{\lambda-2k}, \tag{3.6}$$

$$Q_{\lambda-1}(z) = \sum_{k=0}^{\leq(\lambda-1)/2} q_{\lambda-1,k}z^{\lambda-(2k+1)}. \tag{3.7}$$

The coefficients $p_{\lambda,k}$ and $q_{\lambda-1,k}$ can be successively determined from the recurrence relations

$$(\lambda - \nu + 1)P_{\lambda+2}(z) = P_\lambda(z) - zP_{\lambda+1}(z), \tag{3.8}$$

$$(\lambda - \nu + 1)Q_{\lambda+1}(z) = Q_{\lambda-1}(z) - zQ_\lambda(z), \tag{3.9}$$

with $P_0(z) = 1$, $P_1(z) = z/2\nu$, $Q_{-1}(z) = 0$ and $Q_0(z) = 1/\nu$.

Proof of Theorem 2. If we replace p by $2N + 3$, then the finite sum in (1.1) can be expressed in the form

$$
\begin{aligned}
&D_{2k-1/2}(z\sqrt{2}) \sum_{\lambda=0}^{2N+2} \frac{(2m,\lambda)}{(2z\sqrt{2})^{\lambda}} P_{\lambda}(z\sqrt{2}) \\
&+ D'_{2k-1/2}(z\sqrt{2}) \sum_{\lambda=1}^{2N+2} \frac{(2m,\lambda)}{(2z\sqrt{2})^{\lambda}} Q_{\lambda-1}(z\sqrt{2}).
\end{aligned} \tag{3.10}
$$

Replacing P_λ and $Q_{\lambda-1}$ by their respective sums given in (3.6) and (3.7), and interchanging summation signs, we obtain

$$
\begin{aligned}
W_{k,m}(z^2) = 2^{1/4-k}\sqrt{z} \Bigg\{ & D_{2k-1/2}(z\sqrt{2}) \sum_{l=0}^{N+1} \frac{a_l}{z^{2l}} \\
& + D'_{2k-1/2}(z\sqrt{2}) \sum_{l=0}^{N} \frac{b_l}{z^{2l+1}} + R_{2N+3} \Bigg\},
\end{aligned} \tag{3.11}
$$

where

$$
a_l = \frac{1}{2^l} \sum_{\lambda \geq 2l}^{2N+2} \frac{(2m,\lambda)}{2^{\lambda}} p_{\lambda,l} \quad \text{and} \quad b_l = \frac{1}{2^{l+1/2}} \sum_{\lambda \geq 2l+1}^{2N+2} q_{\lambda-1,l} \frac{(2m,\lambda)}{2^{\lambda}}. \tag{3.12}
$$

The final result (3.4) now follows from (3.1), (3.3) and

$$
D'_{\nu}(z\sqrt{2}) \sim (-\tfrac{1}{2})(z\sqrt{2})^{\nu+1} \exp[-z^2/2] \tag{3.13}
$$

as $z \to \infty$ in $|\arg z| < \frac{3}{4}\pi$.

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