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A Useful Approximation to e^{-t^2}

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Abstract. Using differential approximation, we obtain a remarkably accurate representation of e^{-t^2} as a sum of three exponentials.

1. Introduction. The function e^{-t^2} occurs in many important contexts in mathematics. In some of these, it is quite useful to replace it by an approximation of some type, such as, for example, a Padé approximation. In this note, we wish to exhibit a surprisingly good approximation as a sum of three exponentials. This is obtained using differential approximation, [1]. The approximation obtained here holds for $0 \leqq t \leqq 1$.

2. Differential Approximation. Given a function $k(t)$ for $0 \leqq t \leqq T$, we determine the coefficients $a_1, a_2, \cdots, a_N$ which minimize the quadratic expression

$$J(a_i) = \int_0^T \left[k^{(N)} + \sum_{i=1}^{N} a_i k^{(N-i)} \right]^2 dt, \tag{2.1}$$

where $k^{(i)}$ denotes the ith derivative. We then expect that the solution of the linear differential equation

$$u^{(N)} + a_1 u^{(N-1)} + \cdots + a_N u = 0, \tag{2.2}$$

with suitable boundary conditions, will yield an approximation to $k(t)$. This is a question in stability theory.

The procedure is most useful when N can be taken small. In this case, $N = 3$ and 5 yield excellent results for $k(t) = e^{-t^2}$, as is demonstrated below.

3. Numerical Results. It turns out that good results are obtained by choosing as initial conditions in (2.2): $u^{(i)}(0) = k^{(i)}(0)$, $i = 0, 1, \cdots, N - 1$. The coefficients a_i are listed in the first column of Table 1.

For the case $N = 3$, the calculated values of $u, u', u'', \cdots$ agree to eight figures with the exact values $k, k', k'', \cdots$, respectively. The accuracy is even better for $N = 5$.

If we express the solution of the linear differential equation as a sum of exponentials, we obtain the expression

$$u(t) = \sum_{i=1}^{N} b_i \exp(-\lambda_i t), \tag{3.1}$$

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where b_i and λ_i can have complex values. These values are calculated and listed in Table 1. The numerical values of function $u(t)$ of the above equation at different time intervals ($0 \leqq t \leqq 1$) are listed in Table 2. In the same table, the absolute errors are also shown.

TABLE 1

N	a_i	b_i	λ_i
	2.7403	.7853	.9180
3	7.9511	$.1074 + i\,.1963$	$.9111 + i\,2.334$
	5.7636	$.1074 - i\,.1963$	$.9111 - i\,2.334$
	4.7471	.6509	.9509
	27.9415	$.1795 + i\,.2204$	$.9503 + i\,1.866$
5	62.5129	$.1795 - i\,.2204$	$.9503 - i\,1.866$
	109.1101	$-.0049 + i\,.0163$	$.9478 + i\,3.930$
	68.1498	$-.0049 - i\,.0163$	$.9478 - i\,3.930$

TABLE 2

	$N = 3$		$N = 5$	
Time	*Calculated Value*	*Absolute Error*	*Calculated Value*	*Absolute Error*
.1	.990020	$.30 \times 10^{-4}$	.990049	$.4 \times 10^{-8}$
.3	.913676	$.255 \times 10^{-3}$	.913931	$.2 \times 10^{-6}$
.5	.778679	$.122 \times 10^{-3}$	.778800	$.2 \times 10^{-7}$
.8	.527665	$.372 \times 10^{-3}$	.527292	$.2 \times 10^{-6}$
1.0	.367951	$.72 \times 10^{-4}$	.367879	$.2 \times 10^{-6}$

4. Discussion. If desired, we can improve the accuracy of the approximation by taking the values of $u^{(i)}(0)$, the initial conditions, as parameters, $u^{(i)}(0) = c_i$ and then, by determining these values by the minimization of the quadratic expression,

$$J(c_i) = \int_0^T \left[k(t) - \sum_{i=1}^{N} c_i u_i \right]^2 dt, \tag{4.1}$$

where $u_1, \cdots, u_N$ are N linearly independent solutions of (2.2).

The integrals which arise are evaluated by using the differential equation (2.2) plus the auxiliary equations

$$\frac{dv_{ij}}{dt} = u_i u_j, \quad v_{ij}(0) = 0, \qquad \frac{dw_i}{dt} = u_j k, \quad w_j(0) = 0. \tag{4.2}$$

Then,

$$v_{ij}(T) = \int_0^T u_i u_j \, dt, \qquad w_j(T) = \int_0^T u_j k \, dt. \tag{4.3}$$

The same technique can often be used in the determination of the coefficients a_i

when the function $k(t)$ satisfies a differential equation, linear or nonlinear. In this case, $k' = -t^2 k$, $k(0) = 1$.

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