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A Note Concerning the Two-Step Lax-Wendroff Method in Three Dimensions

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Abstract. The two-step Lax-Wendroff method in three spatial dimensions is discussed and, dealing with its linear stability in the hydrodynamic case, the sufficiency of the von Neumann condition is proved.

In their paper [1], Rubin and Preiser suggest a difference scheme for the conservation law:

$$W_t + \partial f_i/\partial x_i = 0. \tag{1}$$

Their scheme is an extension of Richtmyer's two-step method to three spatial dimensions. In order to deal with the linear stability they take the linearized equation: $W_t + A_i \cdot \partial w/\partial x_i$ (where $A_i = \partial f_i/\partial W$ are taken locally constant), and, in order to get a stability criterion, they compute the amplification matrix:

$$G = I - \tfrac{2}{3} i\lambda[\operatorname{Cos} \xi_1 + \operatorname{Cos} \xi_2 + \operatorname{Cos} \xi_3]M - 2\lambda^2 M^2. \tag{2}$$

Here $\lambda = \Delta t/\Delta x_1 = \Delta t/\Delta x_2 = \Delta t/\Delta x_3$, $\xi_i = k_i \Delta x_i$ (where k_i are the dual variables) and $M = A_1 \operatorname{Sin} \xi_1 + A_2 \operatorname{Sin} \xi_2 + A_3 \operatorname{Sin} \xi_3$.

To prove sufficiency of the von Neumann condition, Rubin and Preiser use a theorem due to Kreiss [3] where the dissipativity of the scheme is assumed. However, it is easy to verify that their scheme is not dissipative because for $\xi = (\pi, 0, 0)$ (so that $|\xi| = \pi$), for example, M is the null matrix and $G = I$ so that its eigenvalues are on the unit circle, as is the case for two dimensions (see [4]).

We give a different proof for the sufficiency of the von Neumann condition but only for the hydrodynamic case. (This is an extension of Richtmyer's proof in [2] to three spatial dimensions.)

In this case, $W = (\rho, \rho u, \rho v, \rho w, E)$, where ρ, E and $\mathbf{V} = (u, v, w)$ are the density, total energy per unit volume and the velocity vector, respectively. We shall make use of the following sufficiency theorem (see [2]): "If G has a complete set of eigenvectors and there exists a constant δ such that $\Delta \geqq \delta > 0$, where Δ^2 is the Gram determinant of the normalized eigenvectors, then the von Neumann condition is sufficient as well as necessary for stability".

Instead of calculating the eigenvectors of G, we shall consider another matrix G' obtained from G by a similarity transformation. We introduce $W' = (\rho, u, v, w, p)$, where p is the pressure, and the transformation is

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$$dW = P \cdot dW', \tag{2.a}$$

$$A_i = PA_i'P^{-1}, \tag{2.b}$$

and so $M = PM'P^{-1}$ and $G = PG'P^{-1}$. This is done because the original A_i are too complicated.

If we compute P from (2.a), we get that

$$\det P = \rho^4\, \partial e/\partial p, \tag{3}$$

where e is the internal energy per unit mass. It turns out that P is triangular so that $\det(P^{-1}) = (\det P)^{-1}$.

Let y_i be the normalized eigenvectors of G (and M), then $\alpha_i P^{-1} y_i$ are the normalized eigenvectors of G' (and M'), where $\alpha_i > 0$ are the normalizing factors.

If we define

$$\Delta_1 = |\det(y_1, y_2, \cdots, y_n)|, \tag{4}$$

$$\Delta_2 = |\det(\alpha_1 P^{-1} y_1, \alpha_2 P^{-1} y_2, \cdots, \alpha_n P^{-1} y_n)|, \tag{5}$$

and $\alpha = \alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n > 0$, then

$$\begin{aligned}\Delta_2 &= |\det[\alpha P^{-1}(y_1, y_2, \cdots, y_n)]| \\ &= \alpha\, |\det P^{-1}| \cdot |\det(y_1, y_2, \cdots, y_n)| \\ &= \alpha\, |\det P^{-1}|\, \Delta_1 = \frac{\alpha}{\rho^4\, \partial e/\partial p}\, \Delta_1.\end{aligned}$$

This is the case because ρ is always bounded away from zero and in the usual fluids the same holds for $\partial e/\partial p$ and consequently for $\det P^{-1}$. Therefore, Δ_1 is bounded away from zero if and only if Δ_2 is.

Rubin and Preiser found M' to be

$$M' = L \begin{bmatrix} u' & \rho \operatorname{Cos} r & \rho \operatorname{Cos} s & \rho \operatorname{Cos} t & 0 \\ 0 & u' & 0 & 0 & 1/\rho \operatorname{Cos} r \\ 0 & 0 & u' & 0 & 1/\rho \operatorname{Cos} s \\ 0 & 0 & 0 & u' & 1/\rho \operatorname{Cos} t \\ 0 & \rho C^2 \operatorname{Cos} r & \rho C^2 \operatorname{Cos} s & \rho C^2 \operatorname{Cos} t & u' \end{bmatrix} \tag{6}$$

where $L = (\operatorname{Sin}^2 \xi_1 + \operatorname{Sin}^2 \xi_2 + \operatorname{Sin}^2 \xi_3)^{1/2}$, $\operatorname{Cos} r = (\operatorname{Sin} \xi_1)/L$, $\operatorname{Cos} s = (\operatorname{Sin} \xi_2)/L$, $\operatorname{Cos} t = (\operatorname{Sin} \xi_3)/L$ and $u' = u \operatorname{Cos} r + v \operatorname{Cos} s + w \operatorname{Cos} t$.

A direct computation of its eigenvectors shows

$$\Delta_2 = \left| \det \begin{bmatrix} 1 & 0 & 0 & -k\rho/C \cdot \operatorname{Cos} r & k\rho/C \cdot \operatorname{Cos} r \\ 0 & -\operatorname{Cos} r \cdot \operatorname{Ctg} s & -\operatorname{Cos} t/\operatorname{Sin} s & k & k \\ 0 & \operatorname{Sin} s & 0 & k \operatorname{Cos} s/\operatorname{Cos} r & k \operatorname{Cos} s/\operatorname{Cos} r \\ 0 & -\operatorname{Cos} t \cdot \operatorname{Ctg} s & \operatorname{Cos} r/\operatorname{Sin} s & k \operatorname{Cos} t/\operatorname{Cos} r & k \operatorname{Cos} t/\operatorname{Cos} r \\ 0 & 0 & 0 & -k\rho C/\operatorname{Cos} r & k\rho C/\operatorname{Cos} r \end{bmatrix} \right| \tag{7}$$

$$= \frac{2\rho C k^2}{\operatorname{Cos}^2 r}.$$

$k^2 = (\mathrm{Cos}^2\, r)/(\rho^2 C^2 + 1 + \rho^2/C^2)$ is a normalizing factor so that finally

$$\Delta_2 = 2\rho C/(\rho^2 C^2 + 1 + \rho^2/C^2). \tag{8}$$

We see that Δ_2, and so Δ_1, is bounded away from zero. Hence, by the sufficiency theorem quoted above, the von Neumann condition, namely $\Delta t/\Delta x \leqq 1/\sqrt{3}(|\mathbf{V}| + C)$, implies linear stability.

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