MATHEMATICS OF COMPUTATION, VOLUME 26, NUMBER 117, JANUARY, 1972

Tridiagonalization of Completely Nonnegative Matrices*

By J. W. Rainey and G. J. Habetler

Abstract. Let $M = [m_{ij}]_{i,j=1}^n$ be completely nonnegative (CNN), i.e., every minor of M is nonnegative. Two methods for reducing the eigenvalue problem for M to that of a CNN, tridiagonal matrix, $T = [t_{ij}]$ ($t_{ij} = 0$ when $|i - j| > 1$), are presented in this paper. In the particular case that M is nonsingular it is shown for one of the methods that there exists a CNN nonsingular S such that $SM = TS$.

1. Introduction. It is well known that if $M = [m_{ij}]_{i,j=1}^n$ is Hermitian, there exists an orthogonal Q such that $QMQ^* = T$ is tridiagonal, i.e., $t_{ij} = 0$ when $|i - j| > 1$. Moreover, for λ (>0) sufficiently large and some nonsingular, diagonal D, $D(T + \lambda I)D^{-1}$ is completely nonnegative (CNN), i.e., every minor of $D(T + \lambda I)D^{-1}$ is nonnegative. (See [2], [3] for a discussion and applications of CNN matrices.) We want to show that an analogous result can be obtained when M is CNN. Namely, we will show that given any arbitrary CNN matrix, M, one can easily construct a CNN tridiagonal matrix, T, which has the same eigenvalues as M. Two methods for obtaining T are described in Section 2, both methods being based upon a result derived in Section 3.

2. Outline of the Methods. (a) *First Method.* If for some k ($2 \leqq k \leqq n - 1$),

$$m_{ij} = 0 \quad (m_{ji} = 0), \qquad i = 1, \cdots, k - 1, j = i + 2, \cdots, n, \tag{2.1}$$

we will say that M is "lower (upper) Hessenberg through its first k rows (columns)." For convenience, we will say that *any* matrix is Hessenberg through its first row or column. A matrix is Hessenberg in the case $k = n - 1$.

In Section 3, we prove the

BASIC LEMMA. *Let M be lower Hessenberg through its first k rows. Then, there exists a CNN matrix, M', which has the same eigenvalues as M and which is lower Hessenberg through its first $k + 1$ rows. If M is nonsingular, then there exists a CNN nonsingular S' such that $S'M = M'S'$.*

By a sequential application of the Basic Lemma, it follows that we can find a CNN lower Hessenberg matrix, H, which has the same eigenvalues as M. We note that if M is nonsingular then $H = S''M(S'')^{-1}$, where S'' is CNN (from, e.g., the Cauchy-Binet theorem [2, I]).

Let P be the matrix obtained by reversing the order of the rows of the $n \times n$ identity, I; trivially, $P^{-1} = P$.

Received January 11, 1971, revised May 24, 1971.

AMS 1970 *subject classifications.* Primary 65F99; Secondary 15A21.

Key words and phrases. Tridiagonalization, tridiagonal matrices, completely nonnegative matrices, Hessenberg matrices.

* Part of this paper appeared in J. W. Rainey's dissertation (Rensselaer, 1967). Research on the dissertation was supported in part by funds from the National Science Foundation under Grant NSF-GP-6339.

Copyright © 1972, American Mathematical Society

Define $\hat{H} = PHP$. $\hat{H}$ is similar to H and therefore has the same eigenvalues as M. $\hat{H}$ is obtained by reversing the order of the rows and columns of H and therefore is *upper* Hessenberg; since the value of a minor is not changed by reversing the order of the rows *and* columns of its array form, $\hat{H}$ must be CNN.

As we indicate in Section 3, a sequential application of our method of proof of the Basic Lemma to $\hat{H}$ maintains the upper Hessenberg form of $\hat{H}$ and therefore yields a CNN tridiagonal matrix, $\hat{T}$. In general, we could take $T = \hat{T}$. In the particular case that M, and therefore $\hat{H}$, is nonsingular, we note as before that there exists a nonsingular CNN S such that $\hat{T} = \hat{S}\hat{H}(\hat{S})^{-1}$; defining

$$\begin{aligned} T &= P\hat{T}P = P\hat{S}\hat{H}(\hat{S})^{-1}P \\ &= P\hat{S}PHP(\hat{S})^{-1}P \\ &= P\hat{S}PS''M(S'')^{-1}P(\hat{S})^{-1}P \\ &= SMS^{-1} \end{aligned}$$

where $S = P\hat{S}PS''$, it is easily verified that T is tridiagonal, CNN, and that S (the product of the CNN matrices, $P\hat{S}P$ and S'') is CNN.

(b) *Second Method.* If, for some k $(2 \leqq k \leqq n - 1)$,

$$m_{ij} = m_{ji} = 0, \qquad i = 1, \cdots, k-1, j = i+2, \cdots, n, \tag{2.2}$$

we will say that M is "tridiagonal through its first k rows and columns." For convenience, we will say that *any* square matrix is tridiagonal through its first row and column. A matrix is tridiagonal in the case $k = n - 1$. We want to prove the

Sequential Lemma. *Let M be tridiagonal through its first k $(<n - 1)$ rows and columns. Then there exists a CNN matrix $\tilde{M}$ which has the same eigenvalues as M and which is tridiagonal through its first $k + 1$ rows and columns.*

Proof. Applying the method of proof of the Basic Lemma to M yields M' which is tridiagonal through its first k rows and columns and lower Hessenberg through its first $k + 1$ rows.

Since every minor of the transpose $(M')^t$ of M' will be the transpose of some minor of M', we note that $(M')^t$ is CNN. Moreover, $(M')^t$ has the same eigenvalues as M, is tridiagonal through its first k rows and columns and is upper Hessenberg through its first $k + 1$ columns. Applying the method of proof of the Basic Lemma to $(M')^t$ would now yield $\tilde{M}$.

The proof of the preceding lemma indicates a method of "sequentially tridiagonalizing" (a term introduced in [1]) M with, as we will show, the desirable property that each intermediate result of the procedure is CNN.

Let $M^{(k)} = [m_{ij}^{(k)}]_{i,j=1}^{n}$ be the $(k - 1)$th result of applying the sequential tridiagonalization procedure to M (in general, $M^{(1)} = M$, $M^{(n-1)} = T$). In analogy with (2.2), we can assume that

$$m_{ij}^{(k)} = m_{ji}^{(k)} = 0, \qquad i = 1, \cdots, k-1, j = i+2, \cdots, n. \tag{2.3}$$

As shown in [4, p. 399 ff.], a measure of the stability of the procedure (but by no means the most important measure) is the growth of the quantities

$$\rho_k = \sum_{j=k+1}^{n} m_{kj}^{(k)} m_{jk}^{(k)}, \tag{2.4}$$

where the ρ_k (see, e.g., [1]) also satisfy

$$(2.5) \qquad \rho_k = t_{k+1,k}t_{k,k+1}, \qquad k = 1, \cdots, n-1.$$

We want to show that the ρ_k cannot become arbitrarily large.

First of all, we note that $M^{(k)}$ $(k > 1)$ is obtained by similarity transformations performed on either $M^{(k-1)}$ or on a "reduced" form of $M^{(k-1)}$; in either case, trace$(M^{(k)})$ = trace$(M^{(k-1)})$ and, therefore, trace(M) = trace$(M^{(k)})$ for all k.

Now, since $M^{(k)}$ is CNN,

$$m_{kk}^{(k)} m_{jj}^{(k)} \geqq m_{kj}^{(k)} m_{jk}^{(k)} \geqq 0, \qquad j \geqq k+1,$$

and, therefore,

$$m_{kk}^{(k)} \sum_{j=k+1}^{n} m_{jj}^{(k)} \geqq \rho_k \geqq 0,$$

or, since trace(M) = trace$(M^{(k)}) = \sum_{j=1}^{n} m_{jj}^{(k)}$,

$$(2.6) \qquad 0 \leqq \rho_k \leqq (\text{trace}(M))^2.$$

By maintaining the CNN property in our procedure, we are assured that the ρ_k remain uniformly bounded with respect to k.

We note that if $M^{(1)}$ is nonsingular, then $M^{(n-1)} = T$ is similar to $M^{(1)}$; letting "$\sim$" indicate similarity, we have, in the notation of the Sequential Lemma, $M \sim M' \sim (M')^t \sim \tilde{M}$ (since any square matrix is similar to its transpose) and by induction, $M^{(1)} \sim T$. Thus, $T = SM^{(1)}S^{-1}$ for some S but the S "constructed" as in the proof of the Basic and Sequential Lemmas is not, in general, CNN. For example, if

$$M = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix},$$

then, following the procedure indicated on the proof of the Sequential Lemma, one obtains

$$T = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 5.5 & .75 \\ 0 & 1 & .5 \end{bmatrix},$$

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .5 & .5 \\ 0 & -1 & 1 \end{bmatrix}.$$

The question of whether or not there exists, in the general case, *some* CNN S such that $T = SMS^{-1}$ remains open.

3. Proof of the Basic Lemma. Let M be CNN and lower Hessenberg through its first k rows but not through its first $k+1$ rows. Then, there exists $p \geqq k+1$ such that

$$(3.1)\qquad M = \left[\begin{array}{ccc|cccc} X & \cdots & X & \overbrace{0}^{p} & & \cdots & 0 \\ \vdots & & \vdots & \vdots & & & \vdots \\ \vdots & & \vdots & 0 & & \cdots & 0 \\ \vdots & & \vdots & u & v & 0 \cdots & 0 \\ \cline{4-7} \vdots & & X & \cdots & \cdots & \cdots & X \\ X & \cdots & \cdots & \cdots & \cdots & \cdots & X \end{array}\right] \begin{array}{l} \\ \\ \}k, \\ \\ \\ \\ \end{array}$$

where the X's indicate possibly nonzero elements, $u = m_{k,p}$ and

$$(3.2)\qquad v = m_{k,p+1} \neq 0.$$

Note. We indicate in (3.1) that $p + 1 < n$ and $k > 1$; whether or not this is true will make no difference in our argument.

We assert that we can verify our primary statement in the lemma by showing that there exists a CNN matrix, say $\hat{M}$, which has precisely the same form as M in (3.1) and the same eigenvalues, but "$\hat{v}$" $= 0$, and then calling on finite induction. We proceed with the proof of the latter.

Consider first the case when $u = m_{kp} = 0$. From the latter assumption, (3.2) and the fact that $u \cdot m_{i,p+1} - v \cdot m_{ip} \geqq 0$ when $i \geqq k$, it follows that the pth column of M must be null. By a similarity transformation involving elementary permutation matrices, one can therefore obtain

$$M' = \begin{bmatrix} M_1' & 0 \\ m_1' & 0 \end{bmatrix},$$

where M_1' is obtained by deleting the pth row and column of M while m_1' is obtained by deleting the pth column of the pth row of M. Now, M_1' would not, in general, be CNN but

$$\hat{M} = \begin{bmatrix} M_1' & 0 \\ 0 & 0 \end{bmatrix}$$

is easily shown to be CNN; moreover, since M' and M are similar, $\hat{M}$ must have the same eigenvalues as M. Finally, from our description of M_1', $\hat{M}$ evidently has the desired form.

Now, suppose that $u \neq 0$. We can, therefore, use u to eliminate v by an "elementary column operation"; in particular, let

$$S^{-1} = I - (v/u)E_pE_{p+1}^t,$$

where I is the $n \times n$ identity and E_i is the ith column of I. We want to show that we may choose

$$(3.3)\qquad \hat{M} = SMS^{-1},$$

where

$$S = I + (v/u)E_pE_{p+1}^t.$$

Since $p \geqq k + 1$, it is evident that $\hat{M}$ has the desired form; it remains now to show that $\hat{M}$ is CNN. Since S is evidently CNN, we can and will verify the latter by showing that $M' = MS^{-1}$ is CNN. Note that if M_i is the ith column of M, then

$$M' = [M_1 \cdots M_p \; M_{p+1} - (v/u)M_p \; M_{p+2} \cdots M_n]. \tag{3.4}$$

In showing that M' is CNN, we assert that we need only consider those minors, of which, say, μ is an example, which satisfy the following conditions:

(a) μ depends upon elements of the $(p+1)$th column of M' but not upon elements of the pth column.

(b) If μ depends upon elements of the first $k-1$ rows of M', then μ depends upon elements of the first $p-1$ columns.

If μ did not satisfy (a), then by inspection of (3.1) and (3.4), μ would be numerically equal to a minor of M; if μ did not satisfy (b), then by inspection, μ depends upon a null row of M. In either of the latter cases, μ would be nonnegative.

For brevity in the following, we introduce the Gantmacher notation: $A\binom{\alpha\ \beta\ \cdots}{a\ b\ \cdots}$ is that submatrix of the matrix A composed of elements from rows $\alpha, \beta, \cdots$ and columns $a, b, \cdots$ while $\bar{A}\binom{\alpha\ \beta\ \cdots}{a\ b\ \cdots}$ is obtained by *deleting* row $\alpha, \beta, \cdots$ and column $a, b, \cdots$ from A. Also, $A[\cdots] = \det\{A(\cdots)\}$ and $\bar{A}[\cdots] = \det\{\bar{A}(\cdots)\}$.

Let μ be a minor of M' satisfying conditions (a) and (b), e.g.,

$$\mu = M'\begin{bmatrix} \alpha\ \beta \cdots\cdots\cdots\cdots\cdots \\ a\ b \cdots c\ p+1\ d \cdots \end{bmatrix}, \tag{3.5}$$

where $\alpha < \beta < \cdots$ and $a < b < \cdots < c < p+1 < d < \cdots$ and $c \neq p$.

Note. Those minors of M' which depend only upon the columns, M_1', $i \geqq p+1$, will be simple special cases of the following.

Now, from (3.4), (3.5) and a well-known determinantal property,

$$\begin{aligned} \mu &= M\begin{bmatrix} \alpha\ \beta \cdots\cdots\cdots\cdots \\ a \cdots c\ p+1\ d \cdots \end{bmatrix} - (v/u)M\begin{bmatrix} \alpha\ \beta \cdots\cdots\cdots \\ a \cdots c\ p\ d \cdots \end{bmatrix}, \\ &= u^{-1}uM\begin{bmatrix} \alpha\ \beta \cdots\cdots\cdots\cdots \\ a \cdots c\ p+1\ d \cdots \end{bmatrix} - vM\begin{bmatrix} \alpha\ \beta \cdots\cdots\cdots \\ a \cdots c\ p\ d \cdots \end{bmatrix}. \end{aligned} \tag{3.6}$$

Let

$$A = M\begin{pmatrix} \alpha\ \beta \cdots \gamma\ k \quad \delta \cdots\cdots \\ a \quad \cdots c\ p\ p+1\ d \cdots \end{pmatrix}, \tag{3.7}$$

where, say, $\gamma < k \leqq \delta$.

Note. If $\alpha > k$, then the first row of A would be composed of elements from the kth row of M; as will be seen, we lose no generality by supposing $k > \alpha$.

For reference, we suppose that $a_{ti} = m_{kp}$. Then, from (3.6) and (3.7),

$$\mu = v^{-1}\left\{a_{ti}\bar{A}\begin{bmatrix} t \\ i \end{bmatrix} - a_{t,i+1}\bar{A}\begin{bmatrix} t \\ i+1 \end{bmatrix}\right\}. \tag{3.8}$$

Thus, we must show that the quantity in brackets is nonnegative.

From (3.1) and (3.7),

$$(3.9) \qquad \begin{array}{r@{}c@{}l}
 & \begin{array}{ccccccc} & & & \overbrace{\;}^{i} & & & \end{array} & \\
A = & \left[\begin{array}{ccc|cccc}
X & \cdots & X & 0 & \cdots & & 0 \\
\vdots & & \vdots & \vdots & & & \vdots \\
\vdots & & \vdots & 0 & \cdots & & 0 \\
\vdots & & \vdots & u' & v' & 0 \cdots & 0 \\ \cline{4-7}
\vdots & & X & \cdots & \cdots & \cdots & X \\
X & \cdots & \cdots & \cdots & \cdots & \cdots & X
\end{array}\right] & \begin{array}{l} \\ \\ \}t, \\ \\ \\ \\ \end{array}
\end{array}$$

where $u' = a_{ti}$, $v' = a_{t,i+1}$. Since, with the possible exception of a "repeated" row, A is a submatrix of M, A is evidently CNN. We require two lemmas, the second of which will readily imply that μ, as defined by (3.8), must be nonnegative when $v > 0$ and A is CNN and has the form noted in (3.9).

The following lemma was proved in [3, p. 309]; for completeness, we offer a proof which does not require certain special results derived in [3].

LEMMA 1. *Let A be CNN. Then, for* $1 \leqq p \leqq n$,

$$(3.10) \qquad (-1)^{p+1} \sum_{i=p}^{n} (-1)^{i+1} a_{1i} \bar{A}\begin{bmatrix} 1 \\ i \end{bmatrix} \geqq 0.$$

Proof. In the case $p = 1$, the left-hand side of (3.10) is just $\det(A)$; in the case $p = n$, the left-hand side reduces to $a_{1n}\bar{A}\binom{1}{n}$. Since A is CNN, (3.1) is evidently valid for these cases.

Assume now that $1 < p < n$. Let s and i be chosen such that $1 \leqq s < p < i \leqq n$ and suppose that, for *all* such pairs (s, i) and *all* k such that $2 \leqq k \leqq n$,

$$\bar{A}\begin{bmatrix} 1 & k \\ s & i \end{bmatrix} = 0.$$

Then, for all $i > p$,

$$\bar{A}\begin{bmatrix} 1 \\ i \end{bmatrix} = \sum_{k=2}^{n} (-1)^{k+s-1} a_{ks} \bar{A}\begin{bmatrix} 1 & k \\ s & i \end{bmatrix} = 0$$

and (3.10) would reduce to the known inequality, $a_{1p}\bar{A}\left[\begin{smallmatrix} 1 \\ p \end{smallmatrix}\right] \geqq 0$.

Assume that for *some* choice of s, i, and k, restricted as above, that

$$(3.11) \qquad \bar{A}\begin{bmatrix} 1 & k \\ s & i \end{bmatrix} \neq 0.$$

Let N be the matrix obtained when the elements $m_{11}, m_{12}, \cdots, m_{1,p-1}$ of A are replaced by zeros. (3.10) is then equivalent to the assertion that

$$(3.12) \qquad (-1)^{p+1} \det(N) \geqq 0.$$

(3.12) is evidently true when $n = 2$; we make the usual inductive hypothesis that (3.12) is valid for all N of dimension less than n. Now, from Sylvester's identity (see, e.g., [2, p. 33]),

$$\det(N)\bar{N}\begin{bmatrix} 1 & k \\ s & i \end{bmatrix} = \bar{N}\begin{bmatrix} 1 \\ s \end{bmatrix}\bar{N}\begin{bmatrix} k \\ i \end{bmatrix} - \bar{N}\begin{bmatrix} 1 \\ i \end{bmatrix}\bar{N}\begin{bmatrix} k \\ s \end{bmatrix},$$

or since all rows, except the first, of A and N are identical and noting (3.11),

$$\det(N) = \left(\bar{A}\begin{bmatrix}1 & k\\ s & i\end{bmatrix}\right)^{-1}\left(\bar{A}\begin{bmatrix}1\\ s\end{bmatrix}\bar{N}\begin{bmatrix}k\\ i\end{bmatrix} - \bar{A}\begin{bmatrix}1\\ i\end{bmatrix}\bar{N}\begin{bmatrix}k\\ s\end{bmatrix}\right). \tag{3.13}$$

Now $\bar{N}\left[\begin{smallmatrix}k\\ i\end{smallmatrix}\right]$ (and $\bar{N}\left[\begin{smallmatrix}k\\ s\end{smallmatrix}\right]$) can be obtained by replacing the first $p - 1$ (and $p - 2$) elements of the first row of $\bar{A}\left[\begin{smallmatrix}k\\ i\end{smallmatrix}\right]$ (and $\bar{A}\left[\begin{smallmatrix}k\\ s\end{smallmatrix}\right]$ with zeros; by the inductive hypothesis

$$(-1)^{p+1}\bar{N}\begin{bmatrix}k\\ i\end{bmatrix} \geqq 0, \qquad (-1)^{(p-1)+1}\bar{N}\begin{bmatrix}k\\ s\end{bmatrix} \geqq 0. \tag{3.14}$$

(3.12), and therefore (3.10), now follow readily from (3.13) and (3.14) and the fact that A is CNN which completes our proof.

The following lemma now generalizes the result of Lemma 1 for the case that A has a form such as in (3.9).

LEMMA 2. *Suppose that A is CNN and that*

$$a_{ij} = 0, \qquad i = 1, \cdots, t-1, j = s, \cdots, n, \tag{3.15}$$

for some s and t satisfying $1 \leqq s, t \leqq n$. *Then, for* $p \geqq s$,

$$(-1)^{t+p}\sum_{j=p}^{n}(-1)^{t+j}a_{tj}\bar{A}\begin{bmatrix}t\\ j\end{bmatrix} \geqq 0. \tag{3.16}$$

Proof. Assume initially that $s > t - 1$. Define

$$r_{ij} = A\begin{bmatrix}1 \cdots t-1\; i+t-2\\ 1 \cdots t-1\; j+t-1\end{bmatrix} \tag{3.17}$$

and let $R = [r_{ij}]_{i,j=1}^{n-(t-1)}$.

Again utilizing Sylvester's identity,

$$R\begin{bmatrix}\epsilon\ \eta \cdots \rho\\ e\ f \cdots g\end{bmatrix} = \Delta^{q-1}A\begin{bmatrix}1 \cdots t-1\; \epsilon+t-1 \cdots \rho+t-1\\ 1 \cdots t-1\; e+t-1 \cdots g+t-1\end{bmatrix}, \tag{3.18}$$

presuming that the latter minor is qth order and

$$\Delta = A\begin{bmatrix}1 \cdots t-1\\ 1 \cdots t-1\end{bmatrix}.$$

Evidently, R is CNN. From Lemma 1, then follows

$$(-1)^{q+1}\sum_{j=q}^{n-(t-1)}(-1)^{1+j}r_{1j}\bar{R}\begin{bmatrix}1\\ j\end{bmatrix} \geqq 0, \tag{3.19}$$

whenever $1 \leqq q \leqq n - (t - 1)$.

Now, from (3.15) and (3.17),

$$r_{1j} = A\begin{bmatrix}1 \cdots t-1 & t\\ 1 \cdots t-1 & j+t-1\end{bmatrix} = a_{t,j+t-1}\Delta,$$

whenever $j \geqq s - (t - 1)$; from (3.18),

$$\bar{R}\begin{bmatrix}1\\ j\end{bmatrix} = \Delta^{n-t-1}\bar{A}\begin{bmatrix}t\\ j+t-1\end{bmatrix}.$$

Utilizing these last two relations and (3.19) yields, after some simplification,

$$\Delta^{n-t}(-1)^{p+t} \sum_{j=p}^{n} (-1)^{t+j} a_{tj} \bar{A}\begin{bmatrix} t \\ j \end{bmatrix} \geqq 0, \tag{3.20}$$

whenever $p \geqq s > t - 1$. If $\Delta > 0$, then (3.20) reduces to (3.16). Suppose, however, that $\Delta = 0$; the inequality

$$\bar{A}\begin{bmatrix} t \\ j \end{bmatrix} \leqq \Delta \bar{A}\begin{bmatrix} 1 & \cdots & t-1 & t \\ 1 & \cdots & t-1 & j \end{bmatrix}, \qquad j > t - 1, \tag{3.21}$$

is a special case of a result due to [2, II, p. 100]. Evidently, $\Delta = 0$ would imply the equality in (3.16) for the case $j \geqq p \geqq s > t - 1$.

Finally, suppose that $s \leqq t - 1$. Then, $A\left[\begin{smallmatrix} 1 & \cdots & s \\ 1 & \cdots & s \end{smallmatrix}\right] = 0$, since $A\left(\begin{smallmatrix} 1 & \cdots & s \\ 1 & \cdots & s \end{smallmatrix}\right)$ has a column of zeros. Then, as in (3.21),

$$\bar{A}\begin{bmatrix} t \\ j \end{bmatrix} \leqq A\begin{bmatrix} 1 & \cdots & s \\ 1 & \cdots & s \end{bmatrix} \bar{A}\begin{bmatrix} 1 & \cdots & s & t \\ 1 & \cdots & s & j \end{bmatrix} = 0, \qquad j > s.$$

Therefore, (3.16) either reduces to an equality (when $p > s$) or to the known inequality, $a_{tp}\bar{A}\left[\begin{smallmatrix} t \\ p \end{smallmatrix}\right] \geqq 0$ (when $p = s$). This completes our proof of the lemma.

From (3.9) and (3.16), then follows

$$\begin{aligned} 0 &\leqq (-1)^{t+1} \sum_{j=1}^{n} (-1)^{t+j} a_{tj} \bar{A}\begin{bmatrix} t \\ j \end{bmatrix}, \\ &= (-1)^{t+i}\left\{(-1)^{t+i} a_{ti} \bar{A}\begin{bmatrix} t \\ i \end{bmatrix} + (-1)^{t+i+1} a_{t,i+1} \bar{A}\begin{bmatrix} t \\ i+1 \end{bmatrix}\right\} \\ &= a_{ti} \bar{A}\begin{bmatrix} t \\ i \end{bmatrix} - a_{t,i+1} \bar{A}\begin{bmatrix} t \\ i+1 \end{bmatrix} \end{aligned}$$

and therefore, from (3.8), $\mu \geqq 0$, which completes our proof of the primary assertion of the Basic Lemma.

Noting that we choose $\hat{M}$ similar to M as long as M does not have a column of zeros, the second assertion of the Basic Lemma is now obvious.

Finally, as in all elementary similarity transformations of the form (3.3), $\hat{M}$ will be upper Hessenberg as long as M is upper Hessenberg.

Department of Mathematics
University of Tulsa
Tulsa, Oklahoma 74104

Department of Mathematics
Rensselaer Polytechnic Institute
Troy, New York 12181

1. F. L. Bauer, "Sequential reduction to tridiagonal form," *J. Soc. Indust. Appl. Math.*, v. 7, 1959, pp. 107–113. MR **20** #6778.

2. F. R. Gantmacher, *The Theory of Matrices*, GITTL, Moscow, 1953; English transl., Vols. 1, 2, Chelsea, New York, 1959. MR **16**, 438; MR **21** #6372c.

3. F. R. Gantmacher & M. G. Kreĭn, *Oscillating Matrices and Kernels and Small Oscillations of Mechanical Systems*, 2nd ed., GITTL, Moscow, 1950; German transl., Akademie-Verlag, Berlin, 1960. MR **14**, 178; MR **22** #5161.

4. J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965. MR **32** #1894.