On Evaluation of Moments of $K_\nu(t)/I_\nu(t)$

By Chih-Bing Ling and Jung Lin

Abstract. This paper presents a method of evaluation of the moments of $K_\nu(t)/I_\nu(t)$. Two pairs of expressions, each consisting of two series, are obtained according to the index being an even or an odd integer. The method is an extension of the method used by Watson. Values are tabulated to 12D for $\nu = 0(1)2$.

In a recent paper [1], Roberts considered the computation of the integral

$$M_k^{(2\nu)} = \int_0^\infty t^k \frac{K_\nu(t)}{I_\nu(t)} \, dt \quad (k \geq 2\nu), \tag{1}$$

where $I_\nu$ and $K_\nu$ are modified Bessel functions. He developed the integral into an 'asymptotic series' suitable for computation when $k$ is a large integer. When $k$ is otherwise a small integer, he integrated numerically the following equivalent integral by using Simpson's rule:

$$M_k^{(2\nu)} = \frac{1}{k + 1} \int_0^\infty t^k \frac{t}{I_\nu(t)} \, dt \quad (k \geq 2\nu). \tag{2}$$

In particular, values are tabulated to 8S for $\nu = 1$ and $k = 2(1)100$.

Some time ago, Watson [2] evaluated the integral in (2) when $k$ is an even integer. He developed the integral into two series by employing a method based on a modification of Plana's summation formula. It is found that this method can be extended to the case when $k$ is an odd integer. Furthermore, it is also found that the same integral can be developed into different series by a modification of the method. Altogether, two pairs of expressions are obtained according to $k$ being an even or an odd integer. It is the purpose of this paper to present such results. Values of the integral are thereby evaluated to 12D for $\nu = 0(1)2$. It is mentioned that a method analogous to the present one was used recently by the authors for the evaluation of two Howland integrals [3].

For convenience, the integral is redefined, together with a factor, as follows:

$$L_k^{(2\nu)} = \frac{2^{k+1}}{\pi(k!)} \int_0^\infty t^k \frac{K_\nu(t)}{I_\nu(t)} \, dt \quad (k \geq 2\nu), \tag{3}$$

so that it tends asymptotically to unity as $k$ tends to infinity. The equivalent integral is

$$L_k^{(2\nu)} = \frac{2^{k+1}}{\pi(k + 1)!} \int_0^\infty \frac{t^k}{I_\nu(t)} \, dt \quad (k \geq 2\nu). \tag{4}$$

The following results are obtained for $k \geq \nu$:

Received September 22, 1971.
AMS 1969 subject classifications. Primary 6525.
Key words and phrases. Moments of $K_\nu(t)/I_\nu(t)$.

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\[ L_{2k}^{(r)} = \frac{2^{2k+1}}{(2k+1)!} \left[ \frac{2^{2r-1}(r!)^2}{\pi} \delta_{r,k} + \alpha \sum_{n=1}^{\infty} \frac{(n\alpha)^{2k}}{I_n^2(n\alpha)} \right. \\
+ (-1)^{r+k} \sum_{n=1}^{\infty} \frac{f_n^{2k-1}}{J_n^2(J_n)} \exp(-a_m) \left. \cdot \left(2k + 1 - a_m(1 + \coth a_m)\right) \right] , \]

\[ L_{2k+1}^{(r)} = \frac{2^{2k+2}}{\pi(2k+2)!} \left[ \frac{2\alpha}{\pi} \sum_{n=1}^{\infty} \frac{(n\alpha)^{2k+1}}{I_n^2(n\alpha)} \left( \text{Si}(n\pi) \right. \\
- (-1)^{r+k} \sum_{n=1}^{\infty} \frac{f_n^{2k}}{J_n^2(J_n)} \sinh a_m \left. \cdot \left(2k + 2 - a_m \coth a_m\right)E_2(a_m) + a_mE_2(a_m) \right) \right] , \]

and

\[ L_{2k}^{(r)} = \frac{2^{2k+1}}{(2k+1)!} \left[ \frac{2^{2r-1}(r!)^2}{\pi} \delta_{r,k} + \alpha \sum_{n=0}^{\infty} \frac{(n\alpha + \frac{1}{2}\alpha)^{2k}}{I_n^2(n\alpha + \frac{1}{2}\alpha)} \right. \\
- (-1)^{r+k} \sum_{n=1}^{\infty} \frac{f_n^{2k+1}}{J_n^2(J_n)} \exp(-a_m) \left. \cdot \left(2k + 1 - a_m(1 + \tanh a_m)\right) \right] , \]

\[ L_{2k+1}^{(r)} = \frac{2^{2k+2}}{\pi(2k+2)!} \left[ \frac{2\alpha}{\pi} \sum_{n=0}^{\infty} \frac{(n\alpha + \frac{1}{2}\alpha)^{2k+1}}{I_n^2(n\alpha + \frac{1}{2}\alpha)} \text{Si}(n\pi + \frac{1}{2}\pi) \right. \\
+ (-1)^{r+k} \sum_{n=1}^{\infty} \frac{f_n^{2k}}{J_n^2(J_n)} \cosh a_m \left. \cdot \left(2 - (2k + 2 - a_m \tanh a_m)E_2(a_m) - a_mE_2(a_m) \right) \right] . \]

The derivation will be described in the Appendix. The first expression in (6) is the one obtained before by Watson while the other three are all new. In these expressions, \( \alpha \) is a positive constant, \( \delta_{r,k} \) is Kronecker delta, \( J_m \) is \( m \)th positive zero of the Bessel function \( J_r(z) \), and \( a_m \) is

\[ a_m = \pi f_m/\alpha. \]

In addition, \( \text{Si} \) is the sine integral, and \( E_2 \) and \( E_3 \) are

\[ \frac{E_2(a)}{E_3(a)} = E_1(a)e^a \pm E_1(-a)e^{-a}, \]

where \( E_1 \) is exponential integral defined by

\[ E_1(a) = \int_a^\infty \frac{e^{-t}}{t} dt. \]

The integral can therefore be computed from one pair of the expressions and checked by the other pair. It is seen that each expression consists of two series. The
constant $\alpha$ is involved on the right of each expression only. This constant can be fixed to suit our convenience. The first series converges rapidly when $\alpha$ is large and the second series when $\alpha$ is small. The first series of the first expression in (5) represents, in fact, the value of the integral given by trapezoidal rule while the first series of the first expression in (6) represents the value given by rectangular rule. In each such series, $\alpha$ stands for the increment. Thus, the second series may be regarded merely as a correction. Its value can be made small by a proper choice of $\alpha$. In the present computation, $\alpha$ will be chosen as 2/5.

The values of $\text{Si}(n\pi/2)$ have been tabulated recently, together with a factor $2/\pi$, by the authors [4] to 25D for $n = 1(1)200$. Further values can be generated readily whenever needed. The computation of $I_\nu(n\pi/2)$ from its series expansion is straightforward. For $\nu = 2$ and $k = 50$, to attain an accuracy of 12D, 160 terms of the first series are needed when $\alpha = 2/5$ while 65 terms are needed when $\alpha = 1$. To attain an accuracy of 8D, the corresponding terms needed are 137 and 56, respectively. The convergence is more rapid when $k$ is smaller. On the other hand, when $\nu$ is smaller, it seems that there is no appreciable effect on the convergence.

The readily accessible values of $j_m$ and $J_{-,2}(j_m)$ are the 10D tables for $\nu = 0$ and 1, [5], [6], and the 8D tables for $\nu = 0(1)20$, [7]. The series expansion of $E_1$ is

\begin{equation}
E_1(\pm a) = -\gamma - \log a - \sum_{m=1}^{\infty} \frac{(\pm a)^m}{m(m!)} \quad (a > 0),
\end{equation}

where $\gamma$ is Euler's constant. Or, when $a$ is large, it is given by the asymptotic series:

\begin{equation}
E_1(a)e^a \sim \frac{1}{a} \left( 1 - \frac{1}{a} + \frac{2!}{a^2} - \frac{3!}{a^3} + \cdots \right),
\end{equation}

\begin{equation}
E_1(-a)e^{-a} \sim -\frac{1}{a} \left( 1 + \frac{1}{a} + \frac{2!}{a^2} + \frac{3!}{a^3} + \cdots \right),
\end{equation}

from which

\begin{equation}
E_2(a) \sim \frac{2}{a^2} \left( 1 + \frac{3!}{a^2} + \frac{5!}{a^4} + \cdots \right),
\end{equation}

\begin{equation}
E_3(a) \sim \frac{2}{a^3} \left( 1 + \frac{2!}{a^2} + \frac{4!}{a^4} + \cdots \right).
\end{equation}

To attain a resulting accuracy of 12D, the existing values of $j_m$ are adequate to compute $E_2(a_m)$ and $E_3(a_m)$ from the asymptotic series in (12) for $m \geq 2$ when $\alpha = 2/5$. For $m = 1$, however, a more accurate value is needed because $E_2$ and $E_3$ are now to be computed through $E_1$ from the series expansion in (10). The accuracy of this value of $j_m$ can be improved readily by using Newton-Raphson method. Generally, two or three terms only are needed to compute the second series when $\alpha = 2/5$. If $\alpha = 1$, however, more terms are needed and the existing values of $j_m$ are adequate only for $m \geq 3$. The Newton-Raphson method used to improve the accuracy of $j_m$ becomes less convenient as $m$ increases.

The computation is carried out on an IBM 360 computer with $\alpha = 2/5$, except that $j_1$ and $E_1(a_1)$ are computed on an IBM 1620 computer. The values obtained from (5) and (6) are in full agreement as anticipated. The results rounded to 12D for $\nu = 0(1)2$ and $k = 0(1)50$ are shown in Table 1. Further values, whenever needed, can be
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computed either from the asymptotic series for the original integral given by Roberts or from a similar series for the equivalent integral given by Brenner and Sonshine [8]. By referring to the former, the series after a slight modification becomes

\[ L_k^{(v)} \sim 1 + c_1^{(v)} \sqrt{k} + c_2^{(v)} \sqrt{k} + c_3^{(v)} \sqrt{k} + \cdots \]

Note that here \( c_n^{(v)} \) is Roberts’ \( b_n \) divided by \( n! \). The first 13 coefficients of \( c_n^{(v)} \) for \( v = 0(1)2 \) are given in Table 2. They are adequate to generate values of the integral to 12D when \( k \geq 50 \).

As mentioned before, when \( v = 1 \), the values of \( M_k^{(1)} \) were tabulated by Roberts to 8S for \( k = 2(1)100 \). When the present values are converted and compared, it is found that Roberts’ values are generally correct, save for a frequent round-off error of one unit in the last digit. When \( v = 0 \), the values of \( 2(2k + 1)M_k^{(0)}/(k!)^2 \) were tabulated recently by Smythe [9] to 8S for \( k = 0(1)83 \). When the present values are converted and compared, it is found that the Smythe’s values generally err in the last digit or occasionally in the seventh digit by one unit. When \( v = 2 \), the values of \( L_k^{(2)} \) were computed before by the first author [10] to 6S for \( k = 4(2)24 \). Comparison shows that the previous values generally err in the fifth or sixth digit, save the first two which err in the third digit.
It is noted that the values of the integral can be used to evaluate allied integrals. For example, consider the following integrals:

\[
S_k^{(r)}(a) = \int_0^\infty t^k K_r(t) \sin(at) \, dt \quad (k + 1 \geq 2\nu),
\]

\[
C_k^{(r)}(a) = \int_0^\infty t^k I_r(t) \cos(at) \, dt \quad (k \geq 2\nu),
\]

\[
I_k^{(r, m, n)}(a) = \int_0^\infty t^k I_r(t) J_m(at) \frac{I_n(at)}{J_n(at)} \, dt \quad (k + m + n \geq 2\nu),
\]

where \( a \) is restricted to be positive and less than unity in the third integral but not restricted in the other integrals. Note that the third integral is in fact a more general integral than the one considered by Sneddon [11, p. 138]. With the aid of the following expansions [12, p. 147],

\[
I_m(at) \frac{I_n(at)}{J_n(at)} = \sum_{p=0}^{\infty} \frac{(-1)^p}{(m + p)! (n + p)!} \left( \frac{m + n + 2p}{p} \right) \left( \frac{a}{2} \right)^{m+n+2p},
\]

and also with those of sine and cosine, the preceding integrals are developed into the following series:

\[
S_k^{(r)}(a) = \frac{\pi(k!)^2}{2^{k+1}} \sum_{p=0}^{\infty} (-1)^p \left( k + 2p + 1 \right) \left( \frac{a}{2} \right)^{2p+1} L_{k+2p+1}^{(r)},
\]

\[
C_k^{(r)}(a) = \frac{\pi(k!)^2}{2^{k+1}} \sum_{p=0}^{\infty} (-1)^p \left( k + 2p \right) \left( \frac{a}{2} \right)^{2p} L_{k+2p}^{(r)},
\]

\[
I_k^{(r, m, n)}(a) = \frac{\pi(k!)^2}{2^{k+1}} \sum_{p=0}^{\infty} \frac{(-1)^p}{k} \left( k + m + n + 2p \right) \left( \frac{a}{m+p} \right)^{m+n+2p} L_{k+m+n+2p}^{(r)}.
\]

A few values thus computed are shown below:

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Appendix. To derive (5), consider the contour integral

\[
\frac{1}{2\pi i} \oint \frac{z^k \, dz}{(z-t) \frac{I_r(z)}{I_r(t)} \sin(\pi z/\alpha)} \quad (k \geq 2\nu),
\]
where the contour is taken around the circle $|z| = R$ through a sequence of values such that the circle never passes through any pole of the integrand. $t$ is any point inside the circle. This integral tends to zero as $R$ tends to infinity. The poles of the integrand are

(A.2) $z = t, \quad z = \pm n\alpha, \quad z = \pm ij_m$.

where $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$. In particular, when $k = 2\nu$, the origin $z = 0$ is also a pole.

It follows from Cauchy's theorem of residues that the sum of residues at all the poles is zero. Consequently, the residue at $z = t$ is

(A.3) $\frac{t^k}{I_r^k(t) \sin(\pi t/\alpha)} = \frac{2^{2r}(\pi \alpha)^{k}}{\pi t} \delta_{2r,k} - \frac{\alpha}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n(n\alpha)^k}{I_r^n(\alpha)} \left( \frac{1}{n\alpha - t} - \frac{(-1)^k}{n\alpha + t} \right) + (-1)^r \sum_{m=1}^{\infty} \frac{(ij_m)^{k-1}}{J_{r+1}(j_m) \sin a_m} \cdot \left[ j_m \left( \frac{1}{(j_m + it)^2} - \frac{(-1)^k}{(j_m - it)^2} \right) \right. \\
\left. - (k + 1 - a_m \coth a_m) \left( \frac{1}{j_m + it} - \frac{(-1)^k}{j_m - it} \right) \right].$

Multiply both sides by $\sin(\pi t/\alpha)$ and integrate with respect to $t$ from zero to infinity.

When $k$ is an even integer, the values of the three integrals on the right are

(A.4) $\int_{0}^{\infty} \frac{t \sin(\pi t/\alpha)}{n^2 \alpha^2 - t^2} \, dt = \frac{\pi}{2} e^{-a_m}$.

When $k$ is an odd integer, they are

(A.5) $\int_{0}^{\infty} \frac{\sin(\pi t/\alpha)}{j_m^2 + t^2} \, dt = \frac{\pi}{2j_m} E_3(a_m), \quad \int_{0}^{\infty} \frac{j_m^2 - t^2 \sin(\pi t/\alpha)}{j_m^2 + t^2} \, dt = \frac{\pi}{2\alpha} E_2(a_m)$.

With these values, the two expressions in (5) are derived.

Next, to derive (6), consider the contour integral

(A.6) $\frac{1}{2\pi i} \oint \frac{z^k \, dz}{(z - t) I_r^k(z) \cos(\pi z/\alpha)} \quad (k \geq 2\nu),$

where the same contour is taken as before. The poles of the integrand are

(A.7) $z = t, \quad z = \pm (n + \frac{1}{2})\alpha, \quad z = \pm ij_m$. 

where \( n = 0, 1, 2, 3, \ldots \) and \( m = 1, 2, 3, \ldots \). Likewise, from Cauchy’s theorem of residues, the residue at \( z = i \) is

\[
\frac{t^k}{J_\nu^k(t) \cos(\pi \nu / \alpha)} = \frac{\alpha}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n(n\alpha + \frac{1}{2}\alpha)^k}{J_n(n\alpha + \frac{1}{2}\alpha)} \left\{ \frac{1}{n\alpha + \frac{1}{2}\alpha - t} + \frac{(-1)^k}{n\alpha + \frac{1}{2}\alpha + t} \right\}
\]

\[+ (-1)^k \sum_{m=1}^{\infty} \frac{t^{j_m - 1}}{J_{j_m+1}(j_m) \cosh a_m}
\]

\[
\left( j_m \left\{ \frac{1}{(j_m + it)^2} + \frac{(-1)^k}{(j_m - it)^2} \right\}
\right.
\]

\[- (k + 1 - a_m \tanh a_m) \left( \frac{1}{j_m + it} + \frac{(-1)^k}{j_m - it} \right) \].

(Multiply both sides by \( \cos(\pi \nu / \alpha) \) and integrate with respect to \( t \) from zero to infinity. According to \( k \) being an even or an odd integer, the values of the three integrals on the right are, respectively,

\[
\int_0^\infty \frac{t \cos(\pi \nu / \alpha) dt}{(n\alpha + \frac{1}{2}\alpha)^2 - t^2} = (-1)^n \frac{\pi}{2(n + 1)\alpha} \sin(\nu \pi + \frac{1}{2}\pi),
\]

(A.9)

\[
\int_0^\infty \frac{t \cos(\pi \nu / \alpha) dt}{j_m^2 + t^2} = \frac{\pi^2}{2\alpha} \exp(-a_m),
\]

\[
\int_0^\infty \frac{t \cos(\pi \nu / \alpha) dt}{(j_m^2 + t^2)^2} \cos \frac{\pi t}{\alpha} dt = \frac{\pi^2}{2\alpha} \exp(-a_m),
\]

and

\[
\int_0^\infty \frac{t \cos(\pi \nu / \alpha) dt}{(n\alpha + \frac{1}{2}\alpha)^2 - t^2} = (-1)^n \frac{\pi}{2(n + 1)\alpha} \sin(\nu \pi + \frac{1}{2}\pi),
\]

(A.10)

\[
\int_0^\infty \frac{t \cos(\pi \nu / \alpha) dt}{j_m^2 + t^2} = \frac{j_m}{2} E_2(a_m),
\]

\[
\int_0^\infty \frac{t \cos(\pi \nu / \alpha) dt}{(j_m^2 + t^2)^2} = \frac{1}{2j_m^2} - \frac{a_m}{4j_m^3} E_3(a_m).
\]

Thence, the two expressions in (6) are derived.

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5. H. T. Davis & W. J. Kirkham, “A new table of zeros of Bessel functions J_0(x) and J_1(x) with corresponding values of J_0(x) and J_1(x),” Bull. Amer. Math. Soc., v. 33, 1927, pp. 760–772; Erratum: J_1(j_0, x), for 98214 read 98314.