On the Solution of Systems of Equations by the Epsilon Algorithm of Wynn

By E. Gekeler

Abstract. The \( \epsilon \)-algorithm has been proposed by Wynn on a number of occasions as a convergence acceleration device for vector sequences; however, little is known concerning its effect upon systems of equations. In this paper, we prove that the algorithm applied to the Picard sequence \( x_{i+1} = F(x_i) \) of an analytic function \( F: \mathbb{R}^n \supset D \rightarrow \mathbb{R}^n \) provides a quadratically convergent iterative method; furthermore, no differentiation of \( F \) is needed. Some examples illustrate the numerical performance of this method and show that convergence can be obtained even when \( F \) is not contractive near the fixed point. A modification of the method is discussed and illustrated.

1. Introduction. The \( \epsilon \)-algorithm is a nonlinear method for accelerating the convergence of sequences; in its simplest form, it is identical with the \( \delta \)-transformation of Aitken [1]. The determinantal formulae upon which it is based were given by Jacobi [6], Schmidt [11], and Shanks [12]; Wynn [13] developed it and examined it thoroughly in connection with various sequences and series [14]–[17]. The \( \epsilon \)-algorithm provides higher (integer) order methods for the computation of a fixed point of an analytic function \( f: \mathbb{C} \supset D \rightarrow \mathbb{C} \) [4]. Using the generalized matrix inverse of Moore [8] and Penrose [9], the method has recently been applied to sequences of matrices and vectors as they arise, for example, in the solution of linear systems of equations [5], [7], [10], [18], [21], [22], [23]. Wynn points out that the algorithm also provides good results in the numerical solution of nonlinear systems [18], [19], [21], [22]. But, until now, nothing is known concerning convergence. In this paper, we examine the behavior of the \( \epsilon \)-algorithm when applied to the Picard sequence of an analytic function \( F: \mathbb{R}^n \supset D \rightarrow \mathbb{R}^n \) with fixed point \( z \). With the help of a theorem of McLeod [7], we show that the algorithm, used in a manner similar to Steffensen's method, is a quadratically convergent iterative method for the computation of \( z \) (compare also Brezinski [2]*).

Because of the complicated recursive relationships, the convergence considered is of local nature, and Landau symbols are used in the proof. A short discussion of numerical properties of the method follows at the end of the paper.

We use certain standard notations: \( i \in \mathbb{N} \) means that \( i \) is a nonnegative integer; lower (upper) case bold face letters denote vectors (matrices); \( ||x|| \) is the Euclidean norm \( (x^*x)^{1/2} \) of the \( n \)-dimensional column vector \( x \in \mathbb{C}^n \); \( O(||x||') \) denotes a vector-valued function of the vector \( x \) whose norm remains bounded as \( ||x|| \rightarrow 0 \) after division by \( ||x||' \); \( O(||x||') \) denotes a real valued function with the same properties.

We also make use of the concept of an analytic function of a vector and of a vector-valued Taylor series. Let \( D \) be an open subset of \( \mathbb{R}^n \), then \( F: \mathbb{R}^n \supset D \rightarrow \mathbb{R}^n \) is called

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analytic if, for every point \( a \in D \), there is an open polycylinder \( P = \{ x \in \mathbb{R}^n : |x_i - a_i| < r_i, 0 < r_i, 1 \leq i \leq n \} \subset D \), such that in \( P \), \( F(x) \) is equal to the sum of an absolutely summable power series in the \( n \) variables \( x_i - a_i \) \((1 \leq i \leq n)\). An analytic function is indefinitely differentiable, and, if the segment joining \( x \) and \( x + y \) is in \( D \), we have, for \( r \in \mathbb{N} \),

\[
F(x + y) = F(x) + \sum_{k=1}^{r-1} \frac{1}{k!} F^{(k)}(x) \cdot y^{(k)} + \left( \int_0^1 \frac{(1-t)^{r-1}}{(r-1)!} F^{(r)}(x + ty) \, dt \right) \cdot y^{(r)},
\]

where \( y^{(k)} \) stands for \((y, y_2, \ldots, y)\) \((k \text{ times})\). For further details, we refer to the famous book of Dieudonné [3].

2. Picard Sequences. We consider some iterative schemes for determining a fixed point \( z \) of the equation \( x = F(x) \). If \( s_p (p \in \mathbb{N}, 0 \leq p) \) is near \( z \), we have, using a Taylor expansion for \( F(z) \),

\[
z = F(s_p) + F'(s_p)(z - s_p) + O(||z - s_p||^2).
\]

Thus, when using the simple iteration scheme

\[
s_{p+1} = F(s_p) \quad (0 \leq p),
\]

we have

\[
z - s_{p+1} = F'(s_p)(z - s_p) + O(||z - s_p||^2).
\]

Hence, the simple scheme (2) is, in general, at best linearly convergent; whether it converges or not depends upon the magnitudes of the eigenvalues of the Jacobian matrices \( F'(s_p) \) \((0 \leq p)\) in the neighbourhood of \( z \). We can, however, devise a quadratically convergent scheme based upon the solution of the linear system

\[
\hat{s}_{p+1} = F(\hat{s}_p) + F'(\hat{s}_p)(s_{p+1} - \hat{s}_p) \quad (0 \leq p)
\]

or

\[
(\mathbf{I} - F'(\hat{s}_p))\hat{s}_{p+1} = F(\hat{s}_p) - F'(\hat{s}_p)s_p \quad (0 \leq p)
\]

for \( s_{p+1} \). For, replacing \( s_p \) in formula (1) by \( \hat{s}_p \), we now have

\[
z - \hat{s}_{p+1} = F'(\hat{s}_p)(z - \hat{s}_{p+1}) + O(||z - \hat{s}_p||^2) \quad (0 \leq p),
\]

i.e.,

\[
(\mathbf{I} - F'(\hat{s}_p))(z - \hat{s}_{p+1}) = O(||z - \hat{s}_p||^2) \quad (0 \leq p)
\]

or, again subject to certain assumptions concerning the eigenvalues of \( F'(x) \) in the neighbourhood of \( z \),

\[
z - \hat{s}_{p+1} = O(||z - \hat{s}_p||^2) \quad (0 \leq p).
\]

The second scheme, although yielding quadratic convergence, involves evaluation of a Jacobian matrix and the solution of a linear system at each stage. However, by use of the \( \epsilon \)-algorithm one can, as we shall show, obtain quadratic convergence without
the computation of the derivatives occurring in the Jacobian matrix, and without the solution of a linear system.

3. The Algorithm. The $\epsilon$-algorithm [13], [22] is a computational procedure in which successive columns of an array \((\epsilon^{(p)})_{0 \leq r \leq q}\) with row index \(p\) are obtained by use of the formula

\[
\epsilon^{(p+1)}_{q+1} = \epsilon^{(p+1)}_{q} + (\epsilon^{(p+1)}_{q} - \epsilon^{(p)}_{q})^{-1} \quad (0 \leq p, 0 \leq q),
\]

starting from the initial conditions

\[
\epsilon^{(p)}_{0} = 0, \quad \epsilon^{(p)}_{p} = s_p \quad (0 \leq p).
\]

If the inverse of a nonzero vector \(x \in \mathbb{C}^n\) is defined, by [8], [9],

\[
x^{-1} = (x^*x)^{-1}x,
\]

then we can apply the algorithm to sequences \(\{s_p\}_{0 \leq p}\) of vectors and have the fundamental theorem [7], [23] which we need later:

**Theorem 1.** Let \(\{s_p\}_{0 \leq p}\) be a sequence of vectors with complex coefficients which satisfy the irreducible linear recursion

\[
\sum_{r=0}^{m} c_r s_{p-r} = \left(\sum_{r=0}^{m} c_r\right)s \quad (0 \leq p),
\]

where \(s\) is fixed and

\[
\sum_{r=0}^{m} c_r \neq 0, \quad c_r \in \mathbb{R}.
\]

If then the elements of the array \(\{\epsilon^{(p)}_{q}\}\) are determined by using (4), (5), and (6), and if all \(\epsilon^{(p)}_{q}\) with \(p + q \leq 2m\) exist, then

\[
\epsilon^{(0)}_{2m} = s.
\]

Following a conjecture of Wynn [24] and Greville [5], Theorem 1 remains true if relations (7), (8) hold for complex scalars only, but this has not yet been proved. In conclusion, we get

**Corollary.** Let \(z\) be the unique solution of the linear system \(x = Ax + c\) with real coefficients and let \(m\) be the degree of the minimal polynomial of the matrix \(A\) for \(y = x_0 - z\). If the $\epsilon$-algorithm is applied to the Picard sequence \(\{x_p\}; x_{p+1} = Ax_p + c\) with vectors \(s_p\) and if all \(\epsilon^{(p)}_{q}\) with \(p + q \leq 2m\) exist, then

\[
\epsilon^{(0)}_{2m} = z.
\]

**Proof.** Let \(p(x) = \sum_{r=0}^{m} a_r x^r\) be the minimal polynomial of \(A\) for \(y\), then

\[
\sum_{r=0}^{m} a_r x^r = \left(\sum_{r=0}^{m} a_r\right)z + \left(\sum_{r=0}^{m} a_r A^r\right)y = \left(\sum_{r=0}^{m} a_r\right)z,
\]

because \(x_p = z + A^p y\) holds. By assumption, we have \(\sum_{r=0}^{m} a_r \neq 0\), since \(1\) is not eigenvalue of \(A\) (the equation \(x = Ax + c\) has a unique solution), and the Corollary results from Theorem 1.
4. The Application of the Epsilon Algorithm to Picard Sequences. The general strategy adopted in deriving our main result is this: we first consider the behaviour of the vectors \( e^{(p)}_q \) (\( p + q \leq 2n \)) derived by means of the \( \epsilon \)-algorithm from the sequence \( s_p = z + A^p y \) (\( 0 \leq p \)), where \( y, z \in \mathbb{R}^n \) and \( A \) is a real \( n \times n \) matrix, for small values of \( ||y|| \) (we know from the above Corollary that, subject to certain conditions, \( e^{(0)} = z \)). We then consider the behaviour of corresponding vectors derived from the sequence \( s_p = s_{p-1} + s_{p} \), where \( s_p = O(||y||^2) \) (\( 0 \leq p \)). Finally, we use these results with \( A = F'(z) \) and

\[
s_{p+1} = F(s_p) = z + F'(z)(s_p - z) + O(||s_p - z||^2) \quad (0 \leq p)
\]
to examine the behaviour of the vectors \( e^{(p)}_q \) produced from this iterative scheme when \( s_0 \) is near a fixed point \( z \) and, in particular, to show that repeated use of the vector \( e^{(p)}_q \) in place of \( s_0 \) results in a quadratically convergent process for determining the fixed point in question. In the sequel, let \( Q_m(A) \subset \mathbb{R}^n \) be the set of vectors \( x \) for which \( m \) is the degree of the minimal polynomial of \( A \).

**Lemma 1.** For a given \( z \), let \( e^{(p)}_q \) be the vectors obtained by means of the \( \epsilon \)-algorithm from the sequence \( \{ s_p; \ s_p = z + A^p y \}_{0 \leq p} \). If there is a neighbourhood \( U \) of \( 0 \) such that all \( t(s^q) \) with \( p + q \leq 2m \) exist for all \( y \in U \setminus Q_m(A) \), then

\[
e^{(p)}_q = z + O(||y||), \quad \text{q even},
\]

\[
e^{(p)}_q = O(||y||^{-1}), \quad \text{q odd},
\]

for \( y \in Q_m(A) \) and \( p + q \leq 2m \).

**Proof.** Let \( m > 0 \), \( p \leq 2m - q \), and \( \Delta_p e^{(p)}_q = e^{(p+1)}_q - e^{(p)}_q \). For \( q = 1 \), we get

\[
(\Delta_p e^{(p)}_q) = A^p (A - I)y = B_p y,
\]

and \( B_p y \neq 0 \) for \( y \in Q_m(A) \), by assumption. Hence,

\[
||e^{(p)}_q|| = ||(y^* B_p y)^{-1} B_p y||
\]

\[
= \frac{1}{||y||} \frac{1}{y^* B_p y} ||B_p y|| \leq \frac{1}{||y||} \frac{||B_p||}{\lambda_{\min}},
\]

where \( 0 < \lambda_{\min} \) is the smallest eigenvalue of \( B_p^* B_p \). Let now \( k \in \mathbb{N} \), \( k < m \), \( y \in Q_m(A) \), and let the statement be true for all \( q \leq 2k \). By assumption, we have \( \Delta_p e^{(p)}_q = O(||y||) \neq 0 \), thus

\[
(\Delta_p e^{(p)}_q)^*(\Delta_p e^{(p)}_q) = O(||y||^2),
\]

\[
e^{(p)}_{2k+1} = e^{(p+1)}_{2k} + [(\Delta_p e^{(p)}_q)^*(\Delta_p e^{(p)}_q)]^{-1} \Delta_p e^{(p)}_q
\]

\[
= O(||y||^{-1}) + O(||y||^{-2}) = O(||y||),
\]

and the assertion of the lemma follows by induction.

**Lemma 2.** Let \( \{ s_p \}_{0 \leq p} \) be a sequence of analytic functions \( s_p(y) = O(||y||^2) \). For a given \( z \), let \( e^{(p)}_q \) be the vectors obtained by means of the \( \epsilon \)-algorithm from the sequence
\[ \varepsilon^{(p)}_q = \varepsilon^{(p)}_q + O(||y||^2), \quad q \text{ even}, \]
\[ \varepsilon^{(p)}_q = \varepsilon^{(p)}_q + O(1), \quad q \text{ odd}, \]
for \( y \in Q_m(A) \) and \( p + q \leq 2m \).

Proof. Let \( m > 0 \) and \( p \geq 2m - q \). For \( q = 1 \), we have \( \Delta_p \varepsilon^{(p)}_0 = \Delta_p \varepsilon^{(p)}_0 + O(||y||^3) \neq 0 \) and \( \Delta_p \varepsilon^{(p)}_0 \neq 0 \) for \( y \in Q_m(A) \), by assumption. Then

\[
\begin{align*}
(\Delta_p \varepsilon^{(p)}_0)^*(\Delta_p \varepsilon^{(p)}_0) &= (\Delta_p \varepsilon^{(p)}_0)^*(\Delta_p \varepsilon^{(p)}_0) + 2(\Delta_p \varepsilon^{(p)}_0)^*O(||y||^3) + O(||y||^4) \\
&= (\Delta_p \varepsilon^{(p)}_0)^*(\Delta_p \varepsilon^{(p)}_0) \left[ 1 + 2 \frac{(\Delta_p \varepsilon^{(p)}_0)^*O(||y||^3)}{(\Delta_p \varepsilon^{(p)}_0)^*(\Delta_p \varepsilon^{(p)}_0)} + \frac{O(||y||^4)}{(\Delta_p \varepsilon^{(p)}_0)^*(\Delta_p \varepsilon^{(p)}_0)} \right].
\end{align*}
\]

\( \Delta_p \varepsilon^{(p)}_0 = O(||y||) \) and hence,

\[
(\Delta_p \varepsilon^{(p)}_0)^*(\Delta_p \varepsilon^{(p)}_0) = (\Delta_p \varepsilon^{(p)}_0)^*(\Delta_p \varepsilon^{(p)}_0)(1 + O(||y||)).
\]

Since \( \Delta_p \varepsilon^{(p)}_0 \) is an analytic function, we get

\[
(\Delta_p \varepsilon^{(p)}_0)(\Delta_p \varepsilon^{(p)}_0) = (\Delta_p \varepsilon^{(p)}_0)(\Delta_p \varepsilon^{(p)}_0)^{-1}[1 + O(||y||)]
\]

and

\[
\varepsilon^{(p)}_1 = \varepsilon^{(p)}_1 + [(\Delta_p \varepsilon^{(p)}_0)^*(\Delta_p \varepsilon^{(p)}_0)]^{-1}O(||y||) + [(\Delta_p \varepsilon^{(p)}_0)^*(\Delta_p \varepsilon^{(p)}_0)]^{-1}[1 + O(||y||)]O(||y||^2)
\]

\[
= \varepsilon^{(p)}_1 + O(1).
\]

Let now \( k \in \mathbb{N}, k < m, y \in Q_m(A) \), and let the statement be true for all \( q \leq 2k \).

By assumption, we have \( \Delta_p \varepsilon^{(p)}_{2k} = \Delta_p \varepsilon^{(p)}_{2k} + O(||y||^3) \neq 0 \) and \( \Delta_p \varepsilon^{(p)}_{2k} \neq 0 \). According to the proof for \( q = 1 \), we get, by use of Lemma 1,

\[
(\Delta_p \varepsilon^{(p)}_{2k})^*(\Delta_p \varepsilon^{(p)}_{2k}) = (\Delta_p \varepsilon^{(p)}_{2k})^*(\Delta_p \varepsilon^{(p)}_{2k})^{-1}[1 + O(||y||)]
\]

and hence,

\[
\varepsilon^{(p)}_{2k+1} = \varepsilon^{(p)}_{2k+1} + O(1).
\]

\( \Delta_p \varepsilon^{(p)}_{2k+1} = \Delta_p \varepsilon^{(p)}_{2k+1} + O(1) \) and \( \Delta_p \varepsilon^{(p)}_{2k+1} \) are equally supposed to be different from zero and, therefore, we get, by use of Lemma 1,

\[
(\Delta_p \varepsilon^{(p)}_{2k+1})^*(\Delta_p \varepsilon^{(p)}_{2k+1})
\]

\[
= (\Delta_p \varepsilon^{(p)}_{2k+1})^*(\Delta_p \varepsilon^{(p)}_{2k+1}) \left[ 1 + 2 \frac{(\Delta_p \varepsilon^{(p)}_{2k+1})^*O(||y||^3)}{(\Delta_p \varepsilon^{(p)}_{2k+1})^*(\Delta_p \varepsilon^{(p)}_{2k+1})} + \frac{O(1)}{(\Delta_p \varepsilon^{(p)}_{2k+1})^*(\Delta_p \varepsilon^{(p)}_{2k+1})} \right]
\]

\[
= (\Delta_p \varepsilon^{(p)}_{2k+1})^*(\Delta_p \varepsilon^{(p)}_{2k+1})(1 + O(||y||)).
\]

\[
\varepsilon^{(p)}_{2k+2} = \varepsilon^{(p)}_{2k+2} + O(||y||^2)
\]

\[
+ [(\Delta_p \varepsilon^{(p)}_{2k+1})^*(\Delta_p \varepsilon^{(p)}_{2k+1})]^{-1}[1 + O(||y||)](\Delta_p \varepsilon^{(p)}_{2k+1} + O(1))
\]

\[
= \varepsilon^{(p)}_{2k+2} + O(||y||^2) + [(\Delta_p \varepsilon^{(p)}_{2k+1})^*(\Delta_p \varepsilon^{(p)}_{2k+1})]^{-1}[1 + O(||y||)]O(1)
\]

\[
= \varepsilon^{(p)}_{2k+2} + O(||y||^2).
\]
In conclusion, we have the following result:

**Theorem 2.** Let $F : \mathbb{R}^n \supset D \rightarrow \mathbb{R}^n$ be an analytic function with fixed point $z \in D$ and let $Q_m(F'(z)) \subset \mathbb{R}^n$ be the set of vectors $x$ for which $m$ is the degree of the minimal polynomial of $F'(z)$. Further, let $\xi^{(p)}_q$ and $\xi^{(q)}_p$ be the vectors obtained by means of the $\epsilon$-algorithm from the sequences

\[
\{s_p ; s_{p+1} = F(s_p)\}_{0 \leq p}, \quad \text{and} \quad \{s_p ; s_p = z + (F'(z))^p (s_0 - z)\}_{0 \leq p},
\]

respectively. Assume that

1. $1$ is not an eigenvalue of $F'(z)$,
2. the vectors $\xi^{(p)}_q$, $\xi^{(q)}_p$, $p + q \leq 2m$, exist for all $s_0$ sufficiently close to $z$ with $s_0 - z \in Q_m(F'(z))$.

Set

\[\xi^{(0)}_{2m} = G(s_0, \ldots, s_{2m}) = H_F(s_0),\]

then the computational procedure

\[x_{i+1} = H_F(x_i) \quad (0 \leq i)\]

is, for $x_0$ sufficiently close to $z$ and $x_0 - z \in Q_m(F'(z))$, a quadratically convergent iterative method for the computation of $z$.

**Proof.** By the corollary and Lemma 2, we have

\[H_F(x_0) = \xi^{(0)}_{2m} = z + O(\|x_0 - z\|^2)\]

for $x_0 - z \in Q_m(F'(z))$.

5. **A Modification of the Method.** When a system of equations $x = F(x)$ of order $n$ is to be solved by the $\epsilon$-algorithm, the way of doing this is normally to put $m = n$. Then, we need, for each step of iteration, $4n^3 + 2n^2$ multiplications, $2n^2 + n$ divisions, $6n^3 - n^2$ additions/subtractions and the computation of $s_p = F(s_{p-1})$ for $1 \leq p \leq 2n$. The computation of the vectors $s_p$ rather quickly produces a characteristic overflow if the eigenvalues of the Jacobian matrix $F'(x)$ are greater in absolute value than unity near the fixed point $z$. This disadvantage can possibly be eliminated by replacing the Picard sequence $s_{p+1} = F(s_p)$ by

\[s_{p+1} = F_\alpha(s_p) = (1 - \alpha)s_{p-1} + \alpha F(s_p) \quad (0 \leq p)\]

with a suitable $\alpha$, $0 < \alpha < 1$; in this way, the rate of growth of the components of the vectors $s_p$ is reduced. If we have, for example, $\rho(F'(z)) = 2$ for the spectral radius $\rho$ of $F'(z)$, we get $\rho(F_\alpha'(z)) = 3/2$ for $\alpha = 1/2$. Those eigenvalues $\lambda$ of $F'(z)$ for which $|\lambda| < 1$ are thereby increased, but they remain smaller than one in absolute value. Apart from this, convergence is slow if the eigenvalues of $F'(x)$ approach one near $z$.

The rounding errors affect the computation severely. Perhaps, it is possible that the numerical properties can be improved if a modification proposed by Wynn [20] is applied. If the eigenvalues $\lambda$ of $F'(x)$ with $|\lambda| < 1$ predominate, we can indicate a modification of the method, by giving up the (theoretic) quadratic convergence, which considerably reduces the amount of work. To achieve this, we replace $2m$ by $2[(m + 1)/2]$ in (9) and obtain for the basic formula of the algorithm

\[\xi^{(0)}_n = G(s_0, \ldots, s_n) = H^{*}_F(s_0),\]

in the case $m = n$ even. We need now, per step of iteration, only
multiplications/divisions,

\[(6n^3 - 2n^2)/8\]

additions/subtractions and the computation of \(s_p = F(s_{p-1})\) for \(1 \leq p \leq n\).

6. Numerical Examples. Let \(F: \mathbb{R}^4 \to \mathbb{R}^4\). In order to illustrate the method of Theorem 2 and its modifications, we consider some systems of quadratic equations \(x = F(x)\) with fixed point \(z = (1, 1, 1, 1)^T\):

(10) \[F(x) = z + F'(z)(x - z) + \frac{1}{2} F''(z)(x - z)^2.\]

For the Taylor series (10), we write briefly

\[F(x) = z + A(x - z) + Q(x - z)\]

and choose for \(A\) (linear) and \(Q\) various mappings. The fixed point \(z\) of the systems given in that manner is computed by means of single-precision arithmetic with ten decimal digits. In detail, let \(P^{(i)}(x) = (p^{(i)}_1(x), \ldots, p^{(i)}_4(x))^T\) and

\[
\begin{align*}
p^{(1)}_1(x) &= -(x_1^2 + x_3 x_4)/2, & p^{(2)}_1(x) &= -x_1^2/4, \\
p^{(1)}_2(x) &= -x_2^2/2, & p^{(2)}_2(x) &= -x_2^2/4, \\
p^{(1)}_3(x) &= -x_3^2/2, & p^{(2)}_3(x) &= -x_3^2/4, \\
p^{(1)}_4(x) &= -(x_4 x_1 + x_2^2)/2, & p^{(2)}_4(x) &= -x_4^2/4.
\end{align*}
\]

Furthermore, let

\[
D = (0.9, 0.8, 0.7, 0.6),
\]

\[
D_2 = (1.5, 0.8, 0.7, 0.6),
\]

\[
D_3 = (2.0, 0.8, 0.7, 0.6)
\]

be diagonal matrices and

\[
U_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}.
\]

We remark that \(U_1\) is orthogonal, whereas \(U_2\) is the ill-conditioned Pascal matrix of order four having an integer-valued inverse. It should be pointed out that

\[
\left( \frac{\partial P^{(i)}(x - z)}{\partial x} \right)_{x = z} = 0 \quad \text{(Matrix)} \quad (j = 1, 2);
\]

hence, choosing \(Q = P^{(i)}\) in eq. (11), we get, indeed, \(F'(z) = A\). Now, if \(A = U_n D U_n^{-1}\) \((l = 1, 2, 3; m = 1, 2)\), then \(D_l\) is the matrix of eigenvalues and \(U_n\) is the matrix of eigenvectors of \(F'(z)\).

In Examples I–VI, \(z\) is computed by the method proposed in Theorem 2.
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</tr>
<tr>
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<td>$6.5 \cdot 10^{-2}$</td>
<td>$4.3 \cdot 10^{-1}$</td>
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<td>$1.6 \cdot 10^{-3}$</td>
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<td>$6.0 \cdot 10^{-3}$</td>
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<td>$2.4 \cdot 10^{-7}$</td>
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<tr>
<td>XI</td>
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<td>$5.1 \cdot 10^{-1}$</td>
<td>$1.1 \cdot 10^{-1}$</td>
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<tr>
<td>XII</td>
<td>1.5</td>
<td>$3.2 \cdot 10^{-1}$</td>
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<td>$2.1 \cdot 10^{-2}$</td>
<td>$1.0 \cdot 10^{-2}$</td>
<td>$9.5 \cdot 10^{-4}$</td>
</tr>
</tbody>
</table>
Example I: \( F'(z) = U_1D_1U_1^{-1}, Q = P^{(1)}, \) initial vector \( x_0 = 2z; \)
Example II: \( F'(z) = U_1D_1U_1^{-1}, Q = P^{(1)}, x_0 = 0; \)
Example III: as Example II but using \( x_0 = 2z; \)
Example IV: \( F'(z) = U_2D_2U_2^{-1}, Q = P^{(2)}, x_0 = 0.5z; \)
Example V: as Example IV but using \( x_0 = 1.5z; \)
Example VI: \( F'(z) = U_1D_1U_1^{-1}, \beta = P_3, x_0 = 2z, \) using the modified Picard sequence \( s_{n+1} = F'(s_n) \) with \( \alpha = 1/2. \)

The Examples VII–XII are the same as Examples I–VI, respectively, but \( z \) is computed using formula (9*) instead of (9).

The above table contains in column \( i (1 \leq i \leq 8) \) the values \( ||x_i - x_{i-1}|| \) (compare Theorem 2) with rounded mantissae; values for which \( ||z - x_i|| < 5.0 \cdot 10^{-9} \) (the process has then terminated) are omitted. Generally speaking, we have found that the algorithm produces better results if the Jacobian matrix of the given system \( \mathbf{x} = F(\mathbf{x}) \) is symmetric. Finally, it should be mentioned that it seems to be impossible at the moment to say more about the error than that it is of quadratic order.

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