Saddle Points of the Complementary Error Function

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Abstract. The first one hundred zeros of the derivative of the function \( w(z) = e^{-z} \text{Erfc}(-iz) \) are given, together with an asymptotic formula for estimating the higher zeros.

1. In a previous paper by the present authors [1], the zeros of the function

\[ w(z) = e^{-z} \text{Erfc}(-iz) \]

were obtained. In this paper, the values of \( z = x + iy \) for which

\[ \frac{dw}{dz} = 0 \]

are given. These points represent singular points of the family of curves

\[ \phi(x, y) \equiv |w| = \text{const} \]

in the \( x\)-\( y \) plane since at such a point the direction \( dy/dx \) of these curves is undefined. As in the case of the zeros of \( w(z) \), the saddle points lie in the lower half-plane and are symmetrically located with respect to the \( y \)-axis. For convenience, we introduce the function \( Y(\rho) = (\rho/\sqrt{2})w(i\rho) \), which satisfies the differential equation

\[ \frac{dY}{d\rho} = 2\rho Y - 1. \]

Thus, at a saddle point, \( \rho = \rho_n \),

\[ 2\rho_n Y(\rho_n) = 1. \]

With the aid of the differential equation (4), we can expand \( Y \) in the vicinity of a saddle point as a Taylor series, viz.,

\[ Y = \frac{1}{2\rho_n} + \frac{1}{2\rho_n} (\rho - \rho_n)^2 + \frac{1}{3}(\rho - \rho_n)^3 + \cdots. \]

Hence

\[ \frac{1}{Y} = 2\rho_n - 2\rho_n(\rho - \rho_n)^2 - \frac{4\rho_n^2}{3}(\rho - \rho_n)^3 + \cdots. \]

Introducing the variable \( t = \rho - 1/2Y \), this may be written

\[ t = (\rho - \rho_n) + \rho_n(\rho - \rho_n)^2 + \frac{2\rho_n^2}{3}(\rho - \rho_n)^3 + \cdots \]

\[ = (\rho - \rho_n) + \rho(\rho - \rho_n)^2 - \left[ 1 - \frac{2\rho_n^2}{3} \right](\rho - \rho_n)^3 + \cdots. \]
Therefore

\[ \rho - \rho_n = t - \rho t^2 + [1 - 8\rho^2/3]t^3 + \cdots , \]

or

\[ \rho_n = \rho - t + \rho t^2 - [1 - 8\rho^2/3]t^3 + \cdots . \]

Equation (3) may also be expressed in terms of \( Y \) as follows:

\[ \rho_n = \frac{1}{2Y} \left[ 1 + t^2 + \frac{4}{3Y} t^3 + \cdots \right]. \]

The above series will converge rapidly if \( \rho \) is close to a saddle point \( \rho_n \). In the next section, an asymptotic approximation to the saddle points is derived which may be used as a first approximation. By computing the corresponding values of \( Y \) and \( t \) and substituting these into Eq. (11), an improved approximation to \( \rho_n \) is obtained. If necessary,* the process may be repeated using the newly computed value of \( \rho \), and continued until convergence is reached. A sample calculation leading to the first saddle point is given at the end of the next section.

2. Asymptotic Approximation to the Saddle Points.  At a saddle point, we have, from Eq. (5), \( 2\rho Y = 1 \) or

\[ w = +i/\pi^{1/2}z. \]

The saddle points are assumed to be of the form \( z = x - iy \), with \( x > 0, \ y > 0 \).

Setting \( w(x + iy) = u + iv \), Eq. (12) is equivalent to

\[ 2e^{y-x}e^{2ixy} - u + iv = i/\pi^{1/2}z. \]

Replacing \( w \) by the first three terms of the continued fraction gives

\[ u - iv = -\frac{i}{\pi^{1/2}} \left[ \frac{z^2 - 1}{z(z^2 - 3/2)} \right], \]

and Eq. (13) becomes

\[ 2e^{y-x}e^{2ixy} \geq -\frac{i}{\pi^{1/2}} \left\{ \frac{1}{z[2z^2 - 3]} \right\}. \]

Since \( \arg(z) = -\pi/4 + \sigma \), it follows that the argument of the right side of (15) is \( \pi/4 - \sigma \). Hence,

\[ 2xy = (2n + \frac{1}{2})\pi + \beta, \]

where \( 0 \leq \beta \leq \pi/2 \) and since, asymptotically, \( x \approx y \), we take, as the limiting value of \( x \) and \( y \),

\[ \lambda = ((n + \frac{1}{8})\pi)^{1/2} \]

and set**

* By computing a sufficient number of additional terms in Eq. (11), only one application would be required.

** For the justification of this form, see [1, Eq. (29)].
From Eq. (16) we have, equating magnitudes,

$$2e^{-4\lambda \alpha - 4\sigma p} \leq \frac{1}{2\sqrt{\pi} (x^2 + y^2)^{3/2}} = \frac{1}{2^{5/2} \sqrt{\pi} [\lambda^2 + \alpha^2 + 2\lambda p]^{3/2}}$$

Hence

$$2e^{-4\lambda \alpha} = 1/2^{5/2} \pi^{1/2} \lambda^3;$$

$$\alpha = \ln(128\pi\lambda^5)/8\lambda.$$

The value of $p$ is determined by equating arguments in Eq. (15). We find, denoting the argument of the right side by $\phi$,

$$\tan 2xy = 1 + 4\alpha^2 - 8\lambda p;$$

$$\tan \phi = 1 - 6\alpha/\lambda + 3/2\lambda^2.$$ 

This gives

$$p = (8(\lambda\alpha)^2 - 12(\lambda\alpha) + 3)/16\lambda^3.$$ 

Thus, the desired asymptotic approximation to three terms is

$$\left\{ \begin{array}{c} x \\ -y \end{array} \right\} = \lambda \pm \frac{1}{8\lambda} \ln(128\pi\lambda^5) + \frac{1}{8} \left[ \ln(128\pi\lambda^5) \right]^2 - \frac{3}{8} \ln(128\pi\lambda^5) + \frac{3}{16\lambda^3}.$$ 

The use of the approximation (25) in conjunction with Eq. (11) is illustrated below for the first saddle point. Equation (17) with $n = 1$ gives

$$\lambda = 1.8799712060$$ 

and this, when substituted into Eq. (25), gives

$$x \approx 2.5332619139, \quad y \approx -1.2321384069.$$ 

The corresponding value of $Y$ is

$$Y = -.0766358650 + .1594090127i.$$ 

Thus

$$t = -.0073085147 + .0144867658i.$$ 

Substituting in Eq. (11) the values of $t$ and $y$ as given by Eqs. (28) and (29), we arrived at the improved values

$$x \approx 2.5471305433, \quad y \approx -1.2251557198,$$ 

the corresponding values of $Y$ and $t$ being

$$Y = -.07667898752 + .1594172691i$$

$$t = .00000137615 - .00000251508i.$$
This leads to the next approximation

\[ x = 2.547128281828 \quad y = -1.22515709595 \]

which is now correct to eleven figures, the error being \( O(t^4) \).

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