REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the indexing system printed in Volume 22, Number 101, January 1968, p. 212.

13 [2.05, 2.30, 2.55, 7].—YUDELL L. LUKE, *On generating Bessel Functions by Use of the Backward Recurrence Formula*, Report ARL 72–0030, Aerospace Research Laboratories, Air Force Systems Command, United Air Force, Wright-Patterson Air Force Base, Ohio, February 1972, iv + 40 pp., 27 cm.

Approximations to the Bessel functions $I_\nu(z)$ and $J_\nu(z)$, which result from J. C. P. Miller's well-known backward recurrence algorithm, are here expressed in terms of hypergeometric functions. It transpires that some of these approximations are identical with certain rational approximations developed elsewhere by the author [1]. The truncation error and the effect of rounding errors are similarly analyzed. Realistic a priori error bounds emerge along with a demonstration that rounding errors in Miller's algorithm are insignificant.

W. G.

1. Y. L. LUKE, *The Special Functions and Their Approximations*, Vols. 1 and 2, Academic Press, New York, 1969.

14 [2.10].—A. H. STROUD, *Approximate Calculation of Multiple Integrals*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1971, xiii + 431 pp., 23 cm. Price $16.50.

The approximate integration of functions of one variable is a subject which today is reasonably well understood, both in its theoretical and practical ramifications, and which is extensively documented in a number of books. The same, unfortunately, cannot be said for integration in higher dimensions. There are several reasons for this. On the theoretical side, one faces the problem of having to cope with an infinite variety of possible regions over which to integrate, in contrast to one dimension, where every connected region is an interval. In addition, there is no theory of orthogonal polynomials in several variables coming to our aid, which would be comparable in simplicity to the well-known one-dimensional theory. On the practical side, one runs up against what R. Bellman refers to as "the curse of dimensionality". The tensor product of a two-point quadrature rule in 100 dimensions calls for $2^{100} \doteq 10^{30}$ function evaluations, a task well beyond the capabilities of even the fastest computers of today. In spite of these formidable difficulties, a good deal of progress has been made, particularly in the last couple of decades. The book under review is the first major attempt of summarizing and codifying current knowledge in the field. The only major omission is S. L. Sobolev's theory of formulas "with a regular boundary layer", which, however, is discussed in a recent survey article by S. Haber [1], and is also expected to be the subject of a forthcoming book by Sobolev.

The book is divided into two parts, entitled "Theory" and "Tables". Part I is concerned with the existence and construction of integration formulas of the form

$$\int_D w(x)f(x)\,dx = \sum_{i=1}^{N} B_i f(\nu_i) + E[f]$$

and with the estimation of the remainder term $E[f]$. Here, D is a domain in n-dimensional Euclidean space and w a given weight function (often identically equal to 1). Part II contains tables of virtually all known formulas of this type and related computer programs.

After an introductory chapter, the discussion begins in Chapter 2 with the construction of product formulas for special n-dimensional regions. The object, thus, is to approximate the desired integral in the form of a Cartesian product of lower-dimensional integration rules, the component rules often being one-dimensional Gaussian rules with suitable weight functions. Among the regions treated are the n-dimensional cube, the n-simplex, n-dimensional cones, the n-sphere and its surface, and the n-dimensional torus. A more general (original) result shows how a formula for a solid star-like n-dimensional region can be obtained from a formula known for its surface. While the use of product formulas is restricted to rather special (though common) regions, nonproduct formulas must be envisaged if one wants to deal with more general, or even arbitrary, regions in n-space. An account of such formulas is given in Chapter 3, one of the longest in the book, and also the most heterogeneous one, methodologically. There are a number of objectives one can pursue and, correspondingly, a number of more or less ad hoc approaches to achieve them. The first (and perhaps easiest) objective is to construct integration formulas, for arbitrary regions and weight functions, having algebraic degree d and requiring not more than $N = (n + d)!/(n!d!)$ function evaluations. It is shown that this can be done, essentially by solving a system of linear algebraic equations. The results are analogues of Newton-Cotes formulas in one dimension. The author then goes on to describe P. J. Davis' procedure for constructing a special class of such formulas, distinguished by having all coefficients B_i positive and all points ν_i contained in the region of interest. The existence of such formulas was proved earlier by Tchakaloff using nonconstructive arguments. The emphasis then shifts to formulas having relatively low degrees and requiring as few points as possible. A great number of these are presented, both for arbitrary and special regions. Their construction involves the solution of certain systems of nonlinear equations by matrix methods, or else is based on the relationship which exists between multivariate orthogonal polynomials and integration formulas. A rudimentary theory (largely due to the author) which explores this relationship is developed in detail. Formulas which have been constructed specifically for two-dimensional regions are also included, notably Radon's fifth-degree seven-point formula. It is proved, quite generally, that a formula of degree $d = 2k$ in n dimensions can have no fewer than $N = (n + k)!/(n!k!)$ points. Similar, but more complicated, results hold for odd degrees. The chapter concludes with a brief discussion of Romberg-type methods for integration over the n-cube. Chapter 4 deals with the extension of formulas to higher dimensions, in particular, with devices, other than product methods, for extending a formula of degree d for the m-cube to a formula of the same degree for the n-cube, where $n > m$. Chapter 5 presents an extensive survey of error estimates. Two kinds of estimates are considered in detail. Both are based on viewing the error $E[f]$ as a linear functional in one function space or another. In the first type of estimate, the functions f are assumed to possess partial derivatives up to a finite order and the spaces considered are typified by the validity of an appropriate Taylor's formula. For such spaces Sard's theory of representing linear functionals can be applied, leading immediately to the desired estimates. The novelty here is that many of the error constants required in these estimates are carefully tabulated and that a number of three-dimensional plots are reproduced which vividly illustrate the behavior of the kernel functions involved. The second type of estimates applies to analytic functions and uses Hilbert space techniques to bound the norm of the error functional. Chapter 6, finally, reviews Monte Carlo and "number-theoretic" methods for integration over the n-cube. These are basically integration rules with all coefficients equal, and points chosen either at random (with uniform distribution over the cube), or in some other fashion designed to reduce the error. This usually involves equidistribution considerations, and it is here where number theory comes in.

Part II begins with a short chapter defining the eighteen regions in n-space for which integration formulas are to be catalogued. The formulas are subsequently listed in Chapter 8, which clearly is the core of the whole work. For each of the eighteen regions there are tabulated formulas of increasing degrees, beginning with degree $d = 1$ and going as high as degree $d = 11$ (and sometimes higher for special two- and three-dimensional regions). Often several formulas are given for the same degree and those, particularly useful in the author's judgment, are identified. With many of the formulas, error constants are tabulated which are useful not only for error estimation, but also for comparison of formulas. References to the original sources are provided with most formulas. Since the points in many of the integration formulas listed are the vertices of certain convex regular n-dimensional polytopes, Chapter 9 provides the coordinates for the respective vertices. Chapter 10, finally, presents a number of FORTRAN programs for selected integration formulas and programs for the evaluation of error constants and Sard's kernel functions. The book concludes with an extensive bibliography containing well over 300 items, two-thirds of them dated 1960 or later. A notable omission is the book by Sobol′ [2], which may have appeared too late for inclusion.

It is an indication of the rapid development of the field that an open question mentioned on p. 100 has since been settled. I. P. Mysovskik͡h and V. I͡a. C͡hernit͡sina [3] showed that regions exist for which the three integrals in (3.12-2) vanish simultaneously. The author's remark (p. 100) "By Theorem 3.15-3 it follows that a fifth-degree formula for R_2 cannot be found with six points" is thereby proved invalid.

The book is written clearly, concisely, and to the point. Typographical errors appear to be very few, and only minor inaccuracies have been noted by the reviewer. On p. 58, e.g., it is misleading to define a hyperplane $\mathcal{H}_{N-1}$ of E_N as "an $(N - 1)$-dimensional subspace of E_N". On page 185 the assumption in the hypercircle inequality (Theorem 5.13-1) is misquoted inasmuch as w should be assumed an element of the hyperplane (5.13-1), not of the hypercircle $\Omega_{r,N}$. To state a conjecture in the form of a theorem (Theorem 4.3-1) is a questionable practice, in the reviewer's opinion. Minor blemishes, such as these, however, do not detract from the enormous value of this monograph as a reference work and systematic exposition of present knowledge on the subject. It will prove an invaluable source of information and a dependable guide to all those who are faced with having to come up with numerical answers to multiple-integration problems.

W. G.

1. S. HABER, "Numerical evaluation of multiple integrals," *SIAM Rev.*, v. 12, 1970, pp. 481–526.
2. I. M. SOBOL', *Multidimensional Quadrature Formulas and Haar Functions,* Izdat. "Nauka", Moscow, 1969. (Russian).
3. I. P. MYSOVSKIKH & V. IA. CHERNITSINA, "Answer to a question of Radon," *Dokl. Akad. Nauk SSSR*, v. 198, 1971, pp. 537–539. (Russian)

15[5, 13.05, 13.15].—G. DUVAUT & J. L. LIONS, *Les Inéquations en Mécanique et en Physique,* Dunod, Paris, 1972, xx + 387 pp., 25 cm. Price 118 Fr.

Important advances in "classical" mathematical physics have been made in the last two decades, due to the consistent application of new techniques in studying partial differential equations. The book under review is a contribution in this direction. For the most part, the text is concerned with providing rigorous proofs of existence and uniqueness theorems for certain classes of partial differential equations of continuum mechanics that have inequalities as boundary conditions. The authors have made some effort to explain the physical meaning of these problems as well as to provide some context for the methods of functional analysis and Sobolev spaces used to solve them. The diverse areas discussed include the equations of plasticity and (linear) elasticity, non-Newtonian (Bingham) fluids, and boundary value problems for Maxwell's equations, among others.

The book consists of seven chapters that can be read independently. In each chapter, various physical problems are formulated in terms of partial differential equations and boundary conditions and then shown to possess "generalized solutions." A reader cannot help but admire the virtuosity of the authors, yet he is left in doubt concerning the deeper aspects and implications of the subject.

A sequel on numerical methods for the problems considered is promised in the near future.

MELVYN S. BERGER

School of Mathematics
The Institute for Advanced Study
Princeton, New Jersey 08540

16[7].—LUDO K. FREVEL, *Evaluation of the Generalized Error Function*, Department of Chemistry, The Johns Hopkins University, Baltimore, Maryland. Ms. of 8 typewritten pp. deposited in the UMT file.

The author tabulates to 5S (unrounded) the "natural" error function

$$϶(x) = \frac{1}{\Gamma(1 + 1/\nu)} \int_0^x e^{-t^\nu} dt$$

for $x = 0(0.01)2$, where the constant ν is determined by the condition that $1/\nu$ is the abscissa of the main minimum of $\Gamma(1 + x)$. Thus, $\nu = 2.166226964$ and $1/\Gamma(1 + 1/\nu) = 1.129173885$ to 9D, as the author correctly states. This choice of ν was apparently motivated by the fact that the error function then exceeds for any specified positive argument x the corresponding value for any other choice of a positive value of ν, as, for example, $\nu = 2$, yielding the normal error function.

The underlying calculations were initially performed on an IBM 370 system and then repeated on a Wang Model 360 calculator. The final computer results were checked to 9S prior to truncating to 5S.

No applications of this unique table are mentioned or suggested.

J. W. W.

17[7].—L. K. Frevel & T. J. Blumer, *Seven-Place Table of Iterated Hyperbolic Tangent*, The Dow Chemical Company, Midland, Michigan, 1972. Ms. of 43 pp. deposited in the UMT file.

The nth iterated hyperbolic tangent is herein tabulated to 7D for $n = 0(0.1)10$ and argument u over the range $u = 0(0.02)3$. All tabular entries were originally calculated to 9D on an IBM 1800 system, prior to rounding to 7D; accuracy is claimed to within a unit in the last tabulated decimal place.

Details of the procedure followed in calculating the table are presented in a three-page introduction, and reference is made to related unpublished tables of iterated functions prepared by the senior author and his associates [1], [2], [3], [4].

A useful figure is included in the text, consisting of an automated plot of the iterated tangent over the tabular range of u and for 30 selected values of n.

J. W. W.

1. L. K. Frevel, J. W. Turley & D. R. Petersen, *Seven-Place Table of Iterated Sine*, The Dow Chemical Company, Midland, Michigan, 1959. [See *Math. Comp.*, v. 14, 1960, p. 76, RMT **2**.]

2. L. K. Frevel & J. W. Turley, *Seven-Place Table of Iterated* $\text{Log}_e(1+x)$, The Dow Chemical Company, Midland, Michigan, 1960. [See *Math. Comp.*, v. 15, 1961, p. 82, RMT **3**.]

3. L. K. Frevel & J. W. Turley, *Tables of Iterated Sine Integral*, The Dow Chemical Company, Midland, Michigan, 1961. [See *Math. Comp.*, v. 16, 1962, p. 119, RMT **8**.]

4. L. K. Frevel & J. W. Turley, *Tables of Iterated Bessel Functions of the First Kind*, The Dow Chemical Company, Midland, Michigan, 1962. [See *Math. Comp.*, v. 17, 1963, pp. 471–472, RMT **81**.]

18[7].—Dušan V. Slavić, "Tables for functions $\Gamma(x)$ and $1/\Gamma(x)$," *Publ. Fac. Elect. Univ. Belgrade* (*Série: Math et Phys.*), No. 357–No. 380, 1971, pp. 69–74.

The two main tables in this publication (No. 372) consist of 30D values of $\Gamma(x)$ and its reciprocal for $x = 1(0.01)2$, as calculated on an IBM 1130 system. A third table gives to the same precision the "principal value" of $\Gamma(-n)$, that is

$$(-1)^n\psi(n + 1)/\Gamma(n + 1),$$

for $n = 0(1)30$.

In the introductory text the author describes the procedure followed in calculating and checking the tables, and he announces the discovery of nine terminal-digit errors in a 16D table of the coefficients of the power series for $1/\Gamma(x)$ that is reproduced in the NBS *Handbook* [1]; however, all these errors have been noted previously [2].

Unfortunately, the great care taken in calculating these tables evidently did not extend to their typesetting and final proofreading. Thus, a comparison of the table of $\Gamma(x)$ with an existing 18D table [3] has revealed three typographical errors; namely, the fourteenth decimal of $\Gamma(1.16)$ should read 8 instead of 3, the fifteenth decimal of $\Gamma(1.28)$ should read 4 instead of 5, and the eleventh decimal of $\Gamma(1.48)$ should read 3 instead of 4. A grave doubt is thereby created as to the complete typographical accuracy of the higher decimals in that table and also with respect to the complete reliability of the other two tables as printed.

J. W. W.

1. M. Abramowitz & I. A. Stegun, Editors, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series, No. 55, U.S. Government Printing Office, Washington, D.C., 1964, p. 256.
2. *Math. Comp.*, v. 20, 1966, p. 641, MTE **399.**
3. British Association for the Advancement of Science, *Mathematical Tables, Vol. I: Circular & Hyperbolic Functions, Exponential & Sine & Cosine Integrals, Factorial Function & Allied Functions, Hermitian Probability Functions*, 3rd ed., Cambridge Univ. Press, 1951, pp. xxxviii + 40.

19[7].—Dušan V. Slavić, "Tables of trigonometric functions (angle in grades)," *Publ. Fac. Elect. Univ. Belgrade* (*Série*: *Math. et Phys.*), No. 357–No. 380, 1971, pp. 75–80.

This paper (No. 373) presents 30D tables of sin x, cos x, tan x, cot x, sec x, and csc x for $x = 0(1^g)100^g$ or, in radians, $x = 0(\pi/200)\pi/2$, calculated on an IBM 1130 system.

A user of these tables may be surprised to see the value of zero assigned to the tabular entries for cot 0 and csc 0; however, this is simply the Cauchy principal value, as the author explains in the introductory text.

Although the author claims accuracy to at least 36D for the computer output, the reviewer has discovered five typographical errors in the printed tables as the result of a comparison with the corresponding 20D tables of Andoyer as reproduced in the NBS *Handbook* [1]. The most conspicuous of these appear in sin 50^g, where the first two decimal figures are incorrect, and in csc 8^g, where the integer part is too large by a unit. The three remaining errors appear in the 20th decimal places of cos 4^g (for 0, read 5), cos 29^g (for 1, read 3), and sec 48^g (for 8, read 2). These corrections have been verified by independent calculations by this reviewer. Of course, there remains the possibility of such errors existing in the last 10 published decimal places, which have not been examined except for a few entries.

J. W. W.

1. M. Abramowitz & I. A. Stegun, Editors, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series, No. 55, U.S. Government Printing Office, Washington, D.C., 1964, Table 4.12, pp. 200–201.

20[7].—HARRY HOCHSTADT, *The Functions of Mathematical Physics*, John Wiley & Sons, Inc., New York, 1971, xi + 322 pp., 24 cm. Price $17.50.

The topics which make up the subject "The Functions of Mathematical Physics," which is also known simply as "The Special Functions," were first studied in the eighteenth and nineteenth centuries by many eminent mathematicians. Their mathematical research on the subject was most often stimulated by the physical applications in which the topics arose. To a large extent, this tendency has persisted to the present day. However, in current times the subject has acquired a mathematical character of its own and has applications in a number of fields far removed from mathematical physics.

The intent of the author of the volume under review is to reflect this historic interplay noted above by developing topics of interest to both applied workers and to mathematicians. He hopes that the selection will enable the reader to consult more specialized treatises and to get new results as needed. Obviously, the author had to exercise considerable judgment in selecting material, as there is a large mass of material available. I find his selection refreshing and informative. However, it should be recognized that no attempt is made to unify various topics as, for example, in the manner of my own work on the subject. (See Y. L. Luke, "*The Special Functions and Their Approximations*," Vols. 1 and 2, Academic Press, 1969; and also *Math. Comp.*, v. 26, 1972, pp. 297–299.)

Chapters 1 and 2 deal with orthogonal polynomials in general and with the classical orthogonal polynomials in particular. The gamma function is the subject of Chapter 3. Chapter 4 is titled "Hypergeometric Functions," but only the ${}_2F_1$ is treated. Legendre functions, a special case of the ${}_2F_1$, are studied in Chapter 5. Chapter 6 treats spherical harmonics in an arbitrary number of dimensions. Confluent hypergeometric functions and Bessel functions are treated in Chapters 7 and 8, respectively. Chapter 9 takes up Hill's equation.

Each chapter contains a set of exercises. There is a subject index but no notation index. The bibliography consists of texts only. Here, some important volumes have been omitted.

Y. L. L.

21[7].—M. M. AGREST & M. S. MAKSIMOV, *Theory of Incomplete Cylindrical Functions and Their Applications*, translated from the Russian by H. E. Fettis, J. W. Goresh and D. A. Lee, Springer-Verlag, New York, 1971, 330 pp., 24 cm. Price $24.50.

A cylinder function is any linear combination of the functions which satisfy Bessel's differential equation. An example is the cylinder function $C_\nu(z) = AJ_\nu(z) + BY_\nu(z)$, where $J_\nu(z)$ and $Y_\nu(z)$ are the familiar Bessel functions of the first and second kind, respectively, and A and B are independent of z. Now, both $J_\nu(z)$ and $Y_\nu(z)$ have a number of integral representations, say of the form $\int_a^b K(x, t)g(t)\,dt$, where a and b are constants independent of x, for example $(a, b) = (0, 1), (0, \pi/2), (1, \infty)$ or $(0, \infty)$. Such integrals are called complete. If either a or b depend on a variable y, then the integral is said to be incomplete. The incomplete function then satisfies a nonhomogeneous differential equation where the homogeneous part is that satisfied by the

Bessel function itself. Incomplete functions are also known as associated Bessel functions. Clearly, there are as many incomplete functions associated with $J_\nu(z)$ as there are integral representations for $J_\nu(z)$ of the kind specified. Remarks similar to the above also hold for $I_\nu(z)$, $K_\nu(z)$ and the Hankel functions. The wordings 'complete' and 'incomplete' are used in a similar fashion for other types of special functions.

There are a number of texts on the special functions that satisfy linear homogeneous differential equations. Except for original works, no textual information on solutions of nonhomogeneous equations exists, except for the rather recent volume by A. W. Babister, *Transcendental Functions Satisfying Nonhomogeneous Linear Differential Equations*, The Macmillan Co., New York, 1967 (see *Math. Comp.*, v. 22, 1968, pp. 223–226). Though the volume under review treats a special differential equation, it is a welcome and valuable addition to the literature, in view of its applicability to numerous problems in mathematical physics and other applied disciplines. Furthermore, there is little overlap with the Babister volume.

The first six chapters deal with various incomplete functions and their mathematical properties, including integral representations, differential and difference equations, series expansions, asymptotic expansions, integrals, etc., i.e., all the properties one normally associates with complete functions. Chapters VII–IX describe numerous applied problems from the fields of wave propagation and diffraction, solid state theory, electromagnetism, atomic and nuclear physics, acoustics, plasma and gas-dynamics and exchange processes between liquid and solid phases which lead to incomplete cylindrical functions.

Finally, Chapter X is a compendium of tables and formulas for evaluation of incomplete cylindrical functions. There are a list of symbols, and author and subject indices. The bibliography of 84 items in the original Russian edition is fairly complete. The translators have added five items to the list, but other references should have been added in both editions.

Y. L. L.

22 [7].—Henry E. Fettis & James C. Caslin, *A Table of the Inverse Sine-Amplitude Function in the Complex Domain*, Report ARL 72-0050, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, April 1972, iv + 174 pp., 28 cm. Copies available from the National Technical Information Service, Springfield, Virginia 22151. Price $3.00.

The Jacobian elliptic functions with complex argument arise in numerous applications, e.g., conformal mapping, and tabular values are available in [1] and [2]. Often, one desires the inverse function. This could be obtained by inverse interpolation in the above tables. However, such a procedure is inconvenient and of doubtful accuracy, especially in some regions where a small change in the variable produces a large change in the function. Charts are available in [1] from which qualitatively correct values of the inverse could be deduced, but no prior explicit tabulation is known.

Consider

$$z = \mathrm{sn}(w, k), \qquad z = a + ib, \qquad w = u + iv,$$

$$u + iv = \mathrm{sn}^{-1}(a + ib) = F(\psi, k),$$

$$a + ib = \sin\psi = \sin(\theta + i\varphi),$$

where $F(\psi, k)$ is the incomplete elliptic integral of the first kind and k is the usual notation for the modulus. Let C, D, E and F stand for certain ranges of the parameters. Thus

$$C : 0(0.1)1; \qquad D : 0.9(0.01)1;$$

$$E : 0.01(0.01)0.1; \qquad F : 0.1(0.1)1.$$

Let K and K' be the complete elliptic integrals of the first kind of modulus k and $k' = (1 - k^2)^{1/2}$, respectively. Then, the tables give 5D values of $u/k + iv/k'$ for

$$k = \sin\theta, \qquad \theta = 5°(5°)85°(1°)89°,$$

and the ranges

$$a = C, b = C; \quad a = D, b = C; \quad a = C, b^{-1} = E; \quad a = C, b^{-1} = F;$$

$$a^{-1} = E, b = C; \quad a^{-1} = F, b = C; \quad a^{-1} = E, b^{-1} = E;$$

$$a^{-1} = F, b^{-1} = F.$$

The headings for each page of the tables were machine printed so that no confusion should arise, provided it is understood that $K = \sin 5$, for example, should read $k = \sin 5°$.

The method of computation and other pertinent formulae are given in the introduction.

Y. L. L.

1. H. E. Fettis & J. C. Caslin, *Elliptic Functions for Complex Arguments*, Report ARL 67-0001, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, January, 1967. See *Math. Comp.*, v. 22, 1968, pp. 230–231.

2. F. M. Henderson, *Elliptic Functions with Complex Arguments*, Univ. of Michigan Press, Ann Arbor, Michigan, 1960. See *Math. Comp.*, v. 15, 1961, pp. 95, 96.

23 [8].—Lai K. Chan, N. N. Chan & E. R. Mead, *Tables for the Best Linear Unbiased Estimate Based on Selected Order Statistics from the Normal, Logistic, Cauchy, and Double Exponential Distribution*, The University of Western Ontario, London, Ontario, 1972. Ms. of 3 typewritten pp. + 1187 computer sheets deposited in the UMT file.

If $X(1) < X(2) < \cdots < X(N)$ represent the order statistics corresponding to a random sample of size N from a population with given probability density function of the form $(1/\sigma)f((x - \mu)/\sigma)$, where μ and σ are location and scale parameters, respectively, then

(1) when $\sigma = \sigma_0$ is known, μ can be estimated by a linear estimate of the form

$$U = a_1 X(n_1) + a_2 X(n_2) + \cdots + a_k X(n_k) - A\sigma_0;$$

(2) when $\mu = \mu_0$ is known, σ can be estimated by a linear estimate of the form

$$S = b_1 X(n_1) + b_2 X(n_2) + \cdots + b_k X(n_k) - B\mu_0;$$

(3) when both μ and σ are unknown, (μ, σ) can be estimated by (U, S) where

$$U = c_1 X(n_1) + c_2 X(n_2) + \cdots + c_k X(n_k),$$

$$S = d_1 X(n_1) + d_2 X(n_2) + \cdots + d_k X(n_k).$$

In all cases, $1 \leqq R_1 \leqq n_1 < \cdots < n_k \leqq N - R_2 \leqq N$, where R_1 and R_2 are, respectively, the number of lower and upper observations that are censored.

In the present tables, $k = 1(1)4$ and the values of the coefficients of $X(n_i)$, A and B, and the ranks $n_1 < \cdots < n_k$ given are such that Lloyd's (1952) best linear unbiased estimate (obtained by the method of generalized least squares) based on the k order statistics $X(n_1) < \cdots < X(n_k)$ has minimum variance (when one parameter is known) or minimum generalized variance (when both are unknown) among the $\binom{N}{k}$ possible choices of the set of k ranks. Also given in the tables are the variances and covariances of the estimates (V(U), V(S), COV(U, S)), the variances of the estimates U and S based on all order statistics in the uncensored portion (V_1, V_2), the relative efficiencies (RE(U) $\equiv V_1/\mathrm{V}(U)$, RE(S) $\equiv V_2/\mathrm{V}(S)$), and the generalized relative efficiencies (GE. RE. $\equiv V_1V_2 - (\mathrm{COV})^2/\mathrm{V}(U)\mathrm{V}(S) - (\mathrm{COV}(U, S))^2$).

The tables include the following distributions: normal distribution, $f(y) = (2\pi)^{-1/2} \exp(-y^2/2)$, for $N = k(1)20$ (194 pages); logistic distribution, $f(y) = [\exp(-y)]/[1 + \exp(-y)]^2$, for $N = k(1)25$ (702 pages); Cauchy distribution, $f(y) = 1/[\pi(1 + y^2)]$, for $N = (k + 4)(1)16(2)20$ (94 pages); and double exponential distribution, $f(y) = (1/2)(\exp -|y|)$, for $N = k(1)20$ (197 pages).

The standard deviation of the logistic distribution is $(\pi/\sqrt{3})\sigma$ and that of the double exponential distribution (also called the Laplace distribution) is 2σ.

Computation of the tables was performed on an IBM 7040 system, with 8D output subsequently rounded to 6D in the final printouts.

More detailed descriptions of these tables and their roles in statistical inference can be found in [1] and [2].

AUTHORS' SUMMARY

1. LAI K. CHAN & N. N. CHAN, "Estimates of the parameters of the double exponential distribution based on selected order statistics," *Bull. Inst. Statist. Res. Training*, v. 3, 1969, pp. 21–40.

2. LAI K. CHAN, N. N. CHAN & E. R. MEAD, "Best linear unbiased estimates of the parameters of the logistic distribution based on selected order statistics," *J. Amer. Statist. Assoc.*, v. 66, 1971, pp. 889–892.

24[9].—BROTHER ALFRED BROUSSEAU, Editor, *Fibonacci and Related Number Theoretic Tables*, Fibonacci Association, St. Mary's College, California, 1972, xii + 151 pp. (spiralbound), 29 cm.

This is a collection of 42 tables which will gladden the hearts of Fibonacci devotees, consisting as it does of 26 tables dealing with sundry matters concerning the Fibonacci numbers (here called "F_n") and their companion sequence (here called "Lucas numbers, L_n"). The other 16 tables deal with other recurring sequences.

The contents of the first 12 tables are as follows:

Tables 1 and 2 give the complete prime decomposition of F_n and L_n for $n \leqq 150$, excerpted from a table of Jarden [1]. Table 1 is correct, except that the editor treats 1 as a prime, and the larger prime factor of F_{71} should read 46165371073. Table 2 is correct, except that the entry 2 for $n = 0$ is omitted, which leads the editor erroneously to underline the entry 2^2 for $n = 3$. The middle digits of L_{108} should be $\cdots 69265847 \cdots$.

Tables 3–5 consist of the squares, cubes, and fourth powers, and their sums, of F_n up to $n = 40$, 35, and 25, respectively.

Tables 6–8 give the same for L_n.

Table 9 contains the prime F_n for $n < 1000$. The last eight digits of F_{131} should be $\cdots 14572169$. The indices n for which F_n is prime were taken from Jarden [1], but contrary to the acknowledgment of the editor, the decimal values of these F_n for $n > 385$ were not, since Jarden's tables extend only to $n = 385$. Their source is consequently obscure.

Table 10 contains the prime L_n for $n < 500$, with the exception of the final entry, L_{353}, which was omitted from the reviewer's copy. This omission is difficult to explain, since the list of values of n used in preparing this table, which includes 353, is given in Jarden. The entry 2 for $n = 0$ is also missing from Table 10.

Table 11 gives the rank of apparition (or rank), here called "entry point," for each prime less than 10^4, as calculated by Wunderlich [2]. In the introduction to this table, the restriction $p \neq 2$ has been omitted both in rule (1) and in the sentence beginning, "If $Z(p)$ is odd, $\cdots$."

Table 12 gives the rank of apparition of all numbers n, $2 \leqq n \leqq 1000$, and also the period of the Fibonacci sequence modulo n.

Among the remaining 30 tables, selected titles are: "Residue cycles of Fibonacci sequences," "Fibonomial coefficients," "Continued fraction expansion of multiples of the golden section ratio," and "Special diagonal sums of Pascal's triangle."

The fact that the tables are not individually numbered, or located by an index with page numbers, makes them difficult to find. Also, the choice of an asterisk to represent the product sign makes the tables with products visually unattractive. This volume well represents the standards of taste and excellence of the Fibonacci Association.

JOHN BRILLHART

Department of Mathematics
Northern Illinois University
De Kalb, Illinois 60115

1. DOV JARDEN, *Recurring Sequences*, 2nd ed., Riveon Lematematika, Jerusalem, 1966. (See *Math. Comp.*, v. 23, 1969, pp. 212–213, RMT **9**.) A new edition is in preparation.

2. MARVIN WUNDERLICH, *Tables of Fibonacci Entry Points*, The Fibonacci Association, San Jose, California, January 1965. (See *Math. Comp.*, v. 20, 1966, pp. 618–619, RMT **87**.)

25 [9].—KATHRYN MILLER, *Solutions of* $\phi(n) = \phi(n + 1)$ *for* $1 \leqq n \leqq 500000$, De Pauw University, Greencastle, Indiana, 1972. Four-page table deposited in the UMT file.

There are 56 solutions of the number-theoretic equation

$$\phi(n) = \phi(n + 1) \tag{1}$$

for $n \leqq (1/2) \cdot 10^6$. A table of these is deposited in the UMT file. For $n \leqq 10^5$, there are 36 solutions, in agreement with [1].

No new solution of

$$\phi(n) = \phi(n + 1) = \phi(n + 2)$$

exists in this range besides the known example $n = 5186$. Except for $n = 1, 3, 5186$ and 5187 all 56 solutions of (1) have n or $n + 1$ divisible by 15 but the next two solutions are

$$n = 525986 = 2 \cdot 181 \cdot 1453, \qquad n + 1 = 3^3 \cdot 7 \cdot 11^2 \cdot 23,$$

and

$$n = 546272 = 2^5 \cdot 43 \cdot 397, \qquad n + 1 = 3^2 \cdot 7 \cdot 13 \cdot 23 \cdot 29.$$

AUTHOR'S SUMMARY

1. M. LAL & P. GILLARD, "On the equation $\phi(n) = \phi(n + k)$," *Math. Comp.*, v. 26, 1972, pp. 579–583.

EDITORIAL NOTE: Equation (1) is necessary but not sufficient for the $\phi(n)$ residue classes prime to n to have the same Abelian group under multiplication (mod n) that the $\phi(n + 1)$ classes have (mod $n + 1$). Of the foregoing 58 solutions, isomorphism is present only in these cases: $n = 1, 3, 15, 104, 495, 975, 22935, 32864, 57584, 131144$, and 491535.

D. S.

26[12].—TORGIL EKMAN & CARL-ERIK FRÖBERG, *Introduction to Algol Programming*, Oxford Univ. Press, London, 1972, 2nd ed. First published in Swedish in 1964, iii + 186 pp., 23 cm. Price $7.95.

Within the hard covers of this 186-page book, there is a concise description of the background and historical development of the ALGOL language, together with a well-arranged comprehensive methodical treatment of the language itself. At the end of most of the chapters, there are exercises, with answers to the questions supplied at the end of the book.

There is not one superfluous word in this text which reads rather well though, at times, it is a little on the heavy side. The humorous quotations introducing each of the chapters (at least, those written in a language understood by the reviewer) served as a timely relief when the going was a little difficult.

The book is not suitable for novices and neither is it intended to be. It is much more attuned to the undergraduate or graduate student with considerable familiarity with Fortran or PL/I programming, although people in a great many different fields of interest would find the language worthy of attention.

HENRY MULLISH

Courant Institute of Mathematical Sciences
New York University
251 Mercer Street
New York, New York 10012

27[12].—Ward Douglas Maurer, *Programming: An Introduction to Computer Techniques*, Holden-Day, Inc., San Francisco, Calif., 1972, xiii + 335 pp., 24 cm. Price $12.95.

This 335-page, hard-cover book by Maurer is an excellent contribution to the literature on computer programming. It assumes a mild familiarity with Fortran, PL/I, and Algol, but even a beginner would not find the going too rough. In fact, this book is extremely well written, comprehensive and, above all, is an attempt to present the fundamentals of computer languages and techniques in a down-to-earth fashion without sacrificing any content.

Specific computers are mentioned along the way together with their particular characteristics. At the end of each chapter is a set of appropriate exercises and problems followed by a list of references for further study.

This text would be ideal for an advanced undergraduate class in computer science. It should prove to be popular with both student and teacher.

Henry Mullish