MATHEMATICS OF COMPUTATION, VOLUME 27, NUMBER 123, JULY, 1973

More About Four Biquadrates Equal One Biquadrate

By Kermit Rose and Simcha Brudno

Abstract. A computer-generated table of the first 82 numerical solutions of $A^4 + B^4 + C^4 + D^4 = E^4$ is presented. Some regularities are noted.

Solutions of the Diophantine equation $A^4 + B^4 + C^4 + D^4 = E^4$ in least integers have been obtained by several authors [1]–[6]. The following table includes the first 82 consecutive primitive solutions of $A^4 + B^4 + C^4 + D^4 = E^4$. A primitive solution is one for which the GCD of A, B, C, D and E equals 1. The method used to find these numerical solutions combines those of Leech [5], Ward [3] and Brudno [6].

In addition, the authors used the following new restrictions:

(1) The sum of two fourth powers cannot be 7, 8, or 11 (Mod 13);

(2) The sum of two fourth powers cannot be 6, 7, 10, or 11 (Mod 17);

(3) The sum of two fourth powers cannot be 4, 5, 6, 9, 13, 22 or 28 (Mod 29);

(4) The sum of three fourth powers cannot be zero modulo 29 unless each fourth power is zero modulo 29. The same applies to modulo 16 and modulo 5.

Some of these restrictions were used by Lander [4] in his search for solutions to $A^4 = B^4 + C^4 + D^4$.

The table is divided into seven columns where the first three are the A_i, B_i, $C_i = 0$ (Mod 5). The fourth column is the $D_i \neq 0$ (Mod 5); the fifth is the appropriate E_i where $A_i^4 + B_i^4 + C_i^4 + D_i^4 = E_i^4$. The sixth column contains an index for each E_i. The following table explains the index:

Index	*Restriction on D*	*Equivalent Formulations*
0	$E - D = 0 \pmod{5^4}$	$E = P + 625Q$, $D = P - 625Q$
1	$182E - D = 0 \pmod{5^4}$	$E = 24P + 7Q$, $D = 24Q - 7P$
2	$E + D = 0 \pmod{5^4}$	$E = 625P + Q$, $D = 625P - Q$
3	$182E + D = 0 \pmod{5^4}$	$E = 24P - 7Q$, $D = 7P + 24Q$

The last column gives a sequence number of the solution. The first 23 solutions were already given elsewhere (see [4]). The last 6 solutions are not consecutive and therefore the numbering sequence stops at 82.

There is a regularity observed in cases 24 to 29, inclusive. They result in a "double triplet" of the E_i with Δ 60. Furthermore, cases 49 and 63 are related by having the same $P = 616$ and $Q = 225$. The numbers were calculated using a CDC 6400 computer at Florida State University, Tallahassee, Florida, and a 7094 at M. I. T., Cambridge, Massachusetts.

Conjecture. Every Diophantine equation of the form

$$\sum_{i=1}^{p-1} X_i^{p-1} = y^{p-1}, \qquad p \text{ a prime},$$

Received April 26, 1972.

AMS (MOS) subject classifications (1970). Primary 10–04, 10B15.

Key words and phrases. Fourth-order equations, numerical tables.

TABLE 1

A_i B_i C_i D_i E_i Index

A	*B*	*C*	*D*	*E*	Index	*NR*
30	120	315	272	353	2	1
240	340	430	599	651	2	2
2420	710	435	1384	2487	1	3
2365	1190	1130	1432	2501	1	4
2745	1010	850	1546	2829	2	5
2460	2345	2270	3152	3723	2	6
3395	3230	350	1652	3973	2	7
2650	1060	205	4094	4267	1	8
3670	3545	1750	1394	4333	3	9
4250	2840	700	699	4449	0	10
1880	1660	380	4907	4949	3	11
5080	1120	1000	3233	5281	3	12
5055	3910	410	1412	5463	2	13
5400	1770	955	2634	5491	2	14
5400	1680	30	3043	5543	0	15
5150	4355	1810	1354	5729	0	16
5695	4280	2770	542	6167	0	17
5000	885	50	5984	6609	0	18
6185	4790	1490	3468	6801	3	19
5365	2850	1390	6368	7101	3	20
2790	1345	160	7166	7209	2	21
6635	5440	800	3052	7339	3	22
6995	5620	2230	3196	7703	1	23
5670	5500	4450	7123	8373	0	24
7565	5230	4730	4806	8433	1	25
7630	5925	4910	524	8493	3	26
7815	6100	3440	1642	8517	0	27
8230	2905	1050	5236	8577	3	28
5780	3695	3450	8012	8637	0	29
8570	6180	3285	816	9137	3	30
6435	2870	680	8618	9243	0	31
7820	6935	5800	5192	9431	1	32
8760	6935	1490	1394	9519	0	33
8570	7050	305	5264	9639	0	34
8835	6800	5490	2922	9797	0	35
6485	5660	4840	8864	9877	1	36
8870	8635	1620	2294	10419	0	37
9145	8530	5300	5936	10939	2	38
10490	8635	5300	3556	11681	0	39
11455	6200	4490	1476	11757	3	40

TABLE 1 (*Continued*)
A_i B_i C_i D_i E_i Index

A	*B*	*C*	*D*	*E*	Index	*NR*
8735	8170	1180	10144	12019	0	41
11720	7270	3710	2833	12167	2	42
9360	8655	7480	8862	12259	3	43
8925	4410	3450	11234	12287	1	44
11390	8045	320	7352	12439	3	45
12435	6190	5780	1616	12759	2	46
10310	6870	2935	10678	12771	3	47
12845	5950	2870	5934	13137	1	48
11210	7590	7025	9712	13209	3	49
13040	4975	1700	7896	13521	0	50
12035	3610	3440	10738	13637	2	51
13410	6420	1275	8278	14029	1	52
13740	7920	6660	3929	14297	1	53
14405	2630	210	34	14409	0	54
13355	8010	1530	9498	14489	1	55
13900	2040	1920	9219	14531	2	56
13760	10245	800	4682	14751	1	57
14815	8940	4250	2512	15309	3	58
11110	6800	3890	14579	15829	0	59
11815	5640	2880	14598	16027	2	60
15780	4790	4140	7701	16049	2	61
15940	6670	5430	137	16113	2	62
14320	13110	2275	1088	16359	1	63
14890	8830	1220	12107	16643	2	64
15160	11015	10850	412	16891	1	65
11810	2350	1845	15776	16893	1	66
15375	11050	6690	11658	17381	3	67
13060	8495	1220	15644	17519	0	68
16405	6500	950	11896	17521	0	69
16215	12850	5450	1802	17661	1	70
10660	3235	3220	17068	17693	0	71
17320	9860	1945	7256	17881	0	72
17510	8340	2760	9423	18077	2	73
16805	13660	5270	5898	18477	2	74
15365	12430	11410	12668	18701	3	75
16560	8355	610	15906	19483	1	76
13940	9305	4460	17726	19493	1	77
17595	13440	5370	12772	19871	1	78
19255	3090	780	12702	20111	1	79
11980	8975	1090	19244	20131	2	80

TABLE 1 (*Continued*)

A_i B_i C_i D_i E_i Index

A	*B*	*C*	*D*	*E*	Index	*NR*
19670	10030	1880	9579	20253	3	81
19480	7550	1660	12969	20469	0	82
18100	13690	12140	11801	20699	2	
13970	8855	8720	19142	20719	3	
17740	16525	12070	3362	21013	2	
13915	5950	5420	24802	25427	0	
16260	12860	8545	34178	34803	0	
1840	30690	41000	89929	91179	0	

has at least one solution in integers such that $x + y = p^{p-1}$ where x is one of the x_i.

The known examples are

$p = 2$.

$$1^1 = 1^1; \qquad 1 + 1 = 2.$$

$p = 3$.

$$3^2 + 4^2 = 5^2; \qquad 4 + 5 = 3^2.$$

$p = 5$.

$$30^4 + 120^4 + 272^4 + 315^4 = 353^4; \qquad 272 + 353 = 5^4.$$

For $p = 7$ no solution is known. The problem of six integers to the sixth power equal to one to the sixth was attacked [4, pp. 454–455] and a computer search for up to $y = 38314$ was unfortunately negative.

However, as indicated by this conjecture, a possible solution may be found for y greater than 58825 ($= 7^6/2 + 1$).

Mathematics Department
Florida State University
Tallahassee, Florida 32306

The Medical Center
University of Illinois
835 South Wolcott Street
Room E-6, MSA Building
Chicago, Illinois 60612

1. R. NORRIE, *University of St. Andrews 500th Anniversary Memorial Volume,* Edinburgh, 1911.

2. J. O. PATTERSON, "A note on the Diophantine problem of finding four biquadrates whose sum is a biquadrate," *Bull., Amer. Math. Soc.*, v. 48, 1942, pp. 736–737. MR **4,** 33.

3. M. WARD, "Euler's problem on sums of three fourth powers," *Duke Math. J.*, v. 15, 1948, pp. 827–837. MR **10,** 283.

4. L. J. LANDER, T. R. PARKIN & J. L. SELFRIDGE, "A survey of equal sums of like powers," *Math. Comp.*, v. 21, 1967, pp. 446–459. MR **36** #5060.

5. J. LEECH, "On $A^4 + B^4 + C^4 + D^4 = E^4$," *Proc. Cambridge Philos. Soc.*, v. 54, 1958, pp. 554–555. MR **20** #2301.

6. S. BRUDNO, "A further example of $A^4 + B^4 + C^4 + D^4 = E^4$," *Proc. Cambridge Philos. Soc.*, v. 60, 1964, pp. 1027–1028. MR **29** #3429.