Chebyshev Polynomials Corresponding to a Semi-Infinite Interval and an Exponential Weight Factor

By David W. Kammler

Abstract. An algorithm is presented for the computation of the $n$ zeros of the polynomial $q_n$ having the property that $q_n(t) \exp(-t)$ alternates $n$ times, at the maximum value 1, on $[0, +\infty)$. Numerical values of the zeros and extremal points are given for $n \leq 10$.

1. Introduction. Using well-known arguments from the theory of minimax approximation (cf. [2, pp. 28–31]), it can be shown that for each $n = 0, 1, 2, \cdots$ there exists a unique polynomial $q_n$ of degree $n$ and $n + 1$ real numbers $0 = r_n^0 < r_n^1 < \cdots < r_n^n$ such that

$$(1) \quad \max\{|q_n(t) \exp(-t)| : t \geq 0\} = 1,$$

$$(2) \quad q_n(r_n^k) \exp(-r_n^k) = (-1)^{n-k}, \quad k = 0, 1, \cdots, n,$$

i.e., such that $q_n$ is the Chebyshev polynomial of degree $n$ which corresponds to the semi-infinite interval $[0, +\infty)$ and to the weight function $w(t) = \exp(-t)$.

By means of a zero counting argument, we see that whenever $y$ satisfies the differential equation

$$(3) \quad (D + 1)^{n+1}y(t) = 0, \quad t \geq 0, \quad D = d/dt$$

and the normalization condition

$$(4) \quad \max\{|y(t)| : t \geq 0\} \leq 1,$$

then

$$(5) \quad |y(t)| \leq q_n(t) \exp(-t) \quad \text{for} \ t \geq r_n^n,$$

with equality possible only if

$$(6) \quad y(t) = \pm q_n(t) \exp(-t), \quad t \geq 0.$$ 

Moreover, it can be shown [1, Theorem 2] that (5) also holds whenever $y$ is any solution of the more general differential equation

$$(7) \quad [(D - \lambda_0) \cdots (D - \lambda_n)]y(t) = 0 \quad \text{for} \ t \geq 0 \text{ with } -\infty < \lambda_0, \cdots, \lambda_n \leq -1$$
which is subject to (4), again with equality possible only if \( y \) is given by (6). In particular, if \( y \) satisfies (7), then

\[
|y(t)| \leq \max \{|y(s)|; 0 \leq s \leq \tau \}|q_n(t)\exp(-t) \quad \text{for } t \geq \tau
\]

Thus, the familiar "maximum growth" property [3, Theorem 6, p. 51] of the ordinary Chebyshev polynomials (associated with the interval \([-1, 1]\) and the weight function \(w(t) = 1\)) corresponds to the "minimal decay rate" (8) for any transient satisfying (7).

By using (8) and the particular function \( y(t) = t^n \exp(-t), \ t \geq 0 \), which takes its maximum at \( t = n \), we conclude that \( \tau \geq n \), \( n = 1, 2, \ldots \). No simple upper bound for \( \tau \) (which could replace \( \tau \) in (8)) is presently known, although we conjecture that \( \tau \) \leq 2n for all \( n \) as is certainly the case for \( n \leq 40 \) (as we have verified numerically).

2. Numerical Determination of \( q_n \). Let \( n \geq 1 \) be fixed and let \( z = (z_1, \ldots, z_n) \) with \( 0 < z_1 < \cdots < z_n \) be given estimates of the zeros of \( q_n \). We define

\[
\varphi(z, t) = [(t/z_1 - 1) \cdots (t/z_n - 1)]\exp(-t), \quad t \geq 0,
\]

and seek to adjust the parameters \( z \) so as to level \( \varphi \) and thereby force \( \varphi \) to satisfy the normalization condition (4).

For \( i = 1, \ldots, n \), we let \( t_i(z) \) denote the unique point where \( |\varphi(z, t)| \) takes its maximum on the interval \( (z_i, z_{i+1}) \) (with \( z_{n+1} \) defined to be \(+\infty\)). Given \( z \), we may numerically determine \( t_i(z) \) by using standard rootfinding techniques (e.g., bisection followed by Newton's method) to locate the unique zero of the function

\[
\varphi_i(z, t) = (t - z_i)^{-1} + \cdots + (t - z_n)^{-1} - 1, \quad z_i < t < z_{i+1}
\]

(with the subscript denoting the corresponding partial derivative).

The perturbation \( h(z) = (h_1(z), \ldots, h_n(z)) \) is defined in such a manner that

\[
\varphi(z + h(z), t_i(z)) \approx (-1)^{n-i}, \quad i = 1, \ldots, n
\]

to terms of first order in \( h(z) \), i.e., such that

\[
\varphi(z, t_i(z)) + \sum_{i=1}^{n} \varphi_i(z, t_i(z))h_i(z) = (-1)^{n-i}, \quad i = 1, \ldots, n
\]

Using (9) in (11), we obtain the equivalent system of linear equations

\[
\sum_{i=1}^{n} h_i(z) = \frac{\varphi(z, t_i(z)) - (-1)^{n-i}}{t_i(z)\varphi(z, t_i(z))}, \quad i = 1, \ldots, n,
\]

which may be used to compute \( h(z) \) when \( z \) is given. (Indeed, since any linear combination of the \( n \) functions \( \psi_i(t) = (t - z_i)^{-1}, \ t \neq z_i, \ i = 1, \ldots, n \), can be expressed as the ratio of two polynomials with the numerator having degree at most \( n - 1 \), it follows that no such linear combination can have more than \( n - 1 \) zeros. Thus, the columns of the coefficient matrix in (12) are linearly independent so that (12) uniquely determines \( h(z) \).

This being the case, we may begin with a suitable initial estimate, \( z_i \), and then successively compute
(13) \( z_{\nu+1} = z_\nu + h(z_\nu), \quad \nu = 1, 2, \ldots, \)

in hopes that this sequence will converge to the limit \( \zeta = (\zeta_{n1}, \ldots, \zeta_{nn}) \) where \( \zeta_{n1} < \cdots < \zeta_{nn} \) are the (positive) zeros of \( q_n \). The sequence will certainly converge to \( \zeta \) provided that \( z_1 \) is sufficiently close to \( \zeta \). Indeed, using (9), (10), (12) and the implicit function theorem we see that \( t_i(z) \) and \( h_i(z), i = 1, \ldots, n \), are all continuously differentiable functions of \( z \) in some neighborhood of \( z = \zeta \) so that we may write

\[
(14) \quad h_i(z) = h_i(\zeta) + \sum_{k=1}^{n} \frac{\partial h_i}{\partial z_k}(\zeta)(z_k - \zeta_{nk}) + o(|z - \zeta|), \quad j = 1, \ldots, n.
\]

Since \( h(z) \) corresponds to a perturbation about \( z = \zeta \) and since \( t_i(z) \) is an extreme point of \( \varphi \), we have

\[
h_i(\zeta) = 0, \quad j = 1, \ldots, n,
\]

\[
\varphi_i(\zeta, t_i(\zeta)) = 0, \quad i = 1, \ldots, n,
\]

and, by making use of these identities in the equation which results when (11) is differentiated with respect to \( z_k \), we obtain

\[
\frac{\partial h_i(\zeta)}{\partial z_k} = -\delta_{ik}, \quad j, k = 1, \ldots, n.
\]

Thus, (14) reduces to

\[
(15) \quad h(z) = \zeta - z + o(|z - \zeta|).
\]

Using (13), (15) and considerations of continuity, we conclude that \( |z_\nu| \) converges to \( \zeta \) and that \( |t_i(z_\nu)| \) converges to the \( k \)th extremal point \( \tau_{nk} \) of (2) for \( q_n \) provided that \( z_\nu \) is sufficiently close to \( \zeta \). (A slight extension of the above argument shows that the convergence is quadratic in each case.)

3. Numerical Results. Using the above procedure, we have computed the zeros \( \zeta_{nk} \) and the extremal points \( \tau_{nk} \) for \( n \leq 40 \), and we list our (rounded) results for \( n \leq 10 \) in Table 1. The roots \( \zeta_{nk}, k = 1, \ldots, n \), can be modeled relatively well by

\[
z_1 = .308/n - .026/n^2,
\]

\[
z_2 = z_1 / .111,
\]

\[
z_k = z_{k-1}/[1 - 2.04/k + .34/k^2 - .10/(n + 2 - k)], \quad k = 3, \ldots, n,
\]

and for \( n \leq 40 \) about a half dozen iterations are needed to locate the zeros and extreme points of \( q_n \) to 16 place accuracy when these initial estimates are used. Finally, we note that the leading coefficient, \( a_n \), of \( q_n \) (which corresponds to the leading coefficient \( 2^{1-n} \) for the ordinary \( n \)th order Chebyshev polynomial) appears to decay with \( n \) in such a manner that

\[
a_n = (\zeta_{n1} \cdots \zeta_{nn})^{-1} \approx [\alpha(\alpha + 1/2) \cdots [\alpha + (n - 1)/2)]^{-1},
\]

\[
\alpha = .276, \quad n = 1, 2, \ldots,
\]

with this approximation being good to within about two percent for \( n \leq 40 \).
Table 1
Zeros and Extremal Points for $q_n$

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1. D. W. Kammler, "A minimal decay rate for solutions of stable n-th order homogeneous differential equations with constant coefficients." (To appear.)
