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Chebyshev Polynomials Corresponding to a Semi-Infinite Interval and an Exponential Weight Factor

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Abstract. An algorithm is presented for the computation of the n zeros of the polynomial q_n having the property that $q_n(t) \exp(-t)$ alternates n times, at the maximum value 1, on $[0, +\infty)$. Numerical values of the zeros and extremal points are given for $n \leqq 10$.

1. Introduction. Using well-known arguments from the theory of minimax approximation (cf. [2, pp. 28–31]), it can be shown that for each $n = 0, 1, 2, \cdots$ there exists a unique polynomial q_n of degree n and $n + 1$ real numbers $0 = \tau_{n0} < \tau_{n1} < \cdots < \tau_{nn}$ such that

$$\max\{|q_n(t) \exp(-t)|: t \geqq 0\} = 1, \tag{1}$$

$$q_n(\tau_{nk}) \exp(-\tau_{nk}) = (-1)^{n-k}, \qquad k = 0, 1, \cdots, n, \tag{2}$$

i.e., such that q_n is the Chebyshev polynomial of degree n which corresponds to the semi-infinite interval $[0, +\infty)$ and to the weight function $w(t) = \exp(-t)$.

By means of a zero counting argument, we see that whenever y satisfies the differential equation

$$(D + 1)^{n+1}y(t) = 0, \qquad t \geqq 0,\ D = d/dt \tag{3}$$

and the normalization condition

$$\max\{|y(t)| : t \geqq 0\} \leqq 1, \tag{4}$$

then

$$|y(t)| \leqq q_n(t) \exp(-t) \quad \text{for } t \geqq \tau_{nn}, \tag{5}$$

with equality possible only if

$$y(t) = \pm q_n(t) \exp(-t), \qquad t \geqq 0. \tag{6}$$

Moreover, it can be shown [1, Theorem 2] that (5) also holds whenever y is any solution of the more general differential equation

$$[(D - \lambda_0) \cdots (D - \lambda_n)]y(t) \equiv 0 \quad \text{for } t \geqq 0 \text{ with } -\infty < \lambda_0, \cdots, \lambda_n \leqq -1 \tag{7}$$

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which is subject to (4), again with equality possible only if y is given by (6). In particular, if y satisfies (7), then

$$(8) \qquad |y(t)| \leqq \max\{|y(s)|: 0 \leqq s \leqq \tau_{nn}\} q_n(t) \exp(-t) \quad \text{for } t \geqq \tau_{nn}.$$

Thus, the familiar "maximum growth" property [3, Theorem 6, p. 51] of the ordinary Chebyshev polynomials (associated with the interval $[-1, 1]$ and the weight function $w(t) \equiv 1$) corresponds to the "minimal decay rate" (8) for any transient satisfying (7).

By using (8) and the particular function $y(t) = t^n \exp(-t)$, $t \geqq 0$, which takes its maximum at $t = n$, we conclude that $\tau_{nn} \geqq n$, $n = 1, 2, \cdots$. No simple upper bound for τ_{nn} (which could replace τ_{nn} in (8)) is presently known, although we conjecture that $\tau_{nn} \leqq 2n$ for all n as is certainly the case for $n \leqq 40$ (as we have verified numerically).

2. Numerical Determination of q_n. Let $n \geqq 1$ be fixed and let $\mathbf{z} = (z_1, \cdots, z_n)$ with $0 < z_1 < \cdots < z_n$ be given estimates of the zeros of q_n. We define

$$(9) \qquad \varphi(\mathbf{z}, t) = [(t/z_1 - 1) \cdots (t/z_n - 1)] \exp(-t), \qquad t \geqq 0,$$

and seek to adjust the parameters $\mathbf{z}$ so as to level φ and thereby force φ to satisfy the normalization condition (4).

For $i = 1, \cdots, n$, we let $t_i(\mathbf{z})$ denote the unique point where $|\varphi(\mathbf{z}, -)|$ takes its maximum on the interval (z_i, z_{i+1}) (with z_{n+1} defined to be $+\infty$). Given $\mathbf{z}$, we may numerically determine $t_i(\mathbf{z})$ by using standard rootfinding techniques (e.g., bisection followed by Newton's method) to locate the unique zero of the function

$$(10) \qquad \varphi_t(\mathbf{z}, t)/\varphi(\mathbf{z}, t) = (t - z_1)^{-1} + \cdots + (t - z_n)^{-1} - 1, \qquad z_i < t < z_{i+1}$$

(with the subscript denoting the corresponding partial derivative).

The perturbation $\mathbf{h}(\mathbf{z}) = (h_1(\mathbf{z}), \cdots, h_n(\mathbf{z}))$ is defined in such a manner that

$$\varphi(\mathbf{z} + \mathbf{h}(\mathbf{z}), t_i(\mathbf{z})) \approx (-1)^{n-i}, \qquad i = 1, \cdots, n,$$

to terms of first order in $\mathbf{h}(\mathbf{z})$, i.e., such that

$$(11) \qquad \varphi(\mathbf{z}, t_i(\mathbf{z})) + \sum_{j=1}^{n} \varphi_{z_j}(\mathbf{z}, t_i(\mathbf{z}))h_j(\mathbf{z}) = (-1)^{n-i}, \qquad i = 1, \cdots, n.$$

Using (9) in (11), we obtain the equivalent system of linear equations

$$(12) \qquad \sum_{j=1}^{n} \frac{h_j(\mathbf{z})}{z_j[t_i(\mathbf{z}) - z_j]} = \frac{\varphi(\mathbf{z}, t_i(\mathbf{z})) - (-1)^{n-i}}{t_i(\mathbf{z})\varphi(\mathbf{z}, t_i(\mathbf{z}))}, \qquad i = 1, \cdots, n,$$

which may be used to compute $\mathbf{h}(\mathbf{z})$ when $\mathbf{z}$ is given. (Indeed, since any linear combination of the n functions $\psi_i(t) = (t - z_i)^{-1}$, $t \neq z_i$, $i = 1, \cdots, n$, can be expressed as the ratio of two polynomials with the numerator having degree at most $n - 1$, it follows that no such linear combination can have more than $n - 1$ zeros. Thus, the columns of the coefficient matrix in (12) are linearly independent so that (12) uniquely determines $\mathbf{h}(\mathbf{z})$.)

This being the case, we may begin with a suitable initial estimate, $\mathbf{z}_1$, and then successively compute

$$\mathbf{z}_{\nu+1} = \mathbf{z}_\nu + \mathbf{h}(\mathbf{z}_\nu), \qquad \nu = 1, 2, \cdots, \tag{13}$$

in hopes that this sequence will converge to the limit $\boldsymbol{\zeta} = (\zeta_{n1}, \cdots, \zeta_{nn})$ where $\zeta_{n1} < \cdots < \zeta_{nn}$ are the (positive) zeros of q_n. The sequence will certainly converge to $\boldsymbol{\zeta}$ provided that $\mathbf{z}_1$ is sufficiently close to $\boldsymbol{\zeta}$. Indeed, using (9), (10), (12) and the implicit function theorem we see that $t_i(\mathbf{z})$ and $h_i(\mathbf{z})$, $i = 1, \cdots, n$, are all continuously differentiable functions of $\mathbf{z}$ in some neighborhood of $\mathbf{z} = \boldsymbol{\zeta}$ so that we may write

$$h_j(\mathbf{z}) = h_j(\boldsymbol{\zeta}) + \sum_{k=1}^{n} \frac{\partial h_j}{\partial z_k}(\boldsymbol{\zeta})(z_k - \zeta_{nk}) + o(|\mathbf{z} - \boldsymbol{\zeta}|), \qquad j = 1, \cdots, n. \tag{14}$$

Since $\mathbf{h}(\mathbf{z})$ corresponds to a perturbation about $\mathbf{z} = \boldsymbol{\zeta}$ and since $t_i(\mathbf{z})$ is an extreme point of φ, we have

$$h_j(\boldsymbol{\zeta}) = 0, \qquad j = 1, \cdots, n,$$

$$\varphi_t(\boldsymbol{\zeta}, t_i(\boldsymbol{\zeta})) = 0, \qquad i = 1, \cdots, n,$$

and, by making use of these identities in the equation which results when (11) is differentiated with respect to z_k, we obtain

$$\partial h_j(\boldsymbol{\zeta})/\partial z_k = -\delta_{jk}, \qquad j, k = 1, \cdots, n.$$

Thus, (14) reduces to

$$\mathbf{h}(\mathbf{z}) = \boldsymbol{\zeta} - \mathbf{z} + o(|\mathbf{z} - \boldsymbol{\zeta}|). \tag{15}$$

Using (13), (15) and considerations of continuity, we conclude that $\{\mathbf{z}_\nu\}$ converges to $\boldsymbol{\zeta}$ and that $\{t_k(\mathbf{z}_\nu)\}$ converges to the kth extremal point τ_{nk} of (2) for q_n provided that $\mathbf{z}_1$ is sufficiently close to $\boldsymbol{\zeta}$. (A slight extension of the above argument shows that the convergence is quadratic in each case.)

3. Numerical Results. Using the above procedure, we have computed the zeros ζ_{nk} and the extremal points τ_{nk} for $n \leqq 40$, and we list our (rounded) results for $n \leqq 10$ in Table 1. The roots ζ_{nk}, $k = 1, \cdots, n$, can be modeled relatively well by

$$z_1 = .308/n - .026/n^2,$$

$$z_2 = z_1/.111,$$

$$z_k = z_{k-1}/\{1 - 2.04/k + .34/k^2 - .10/(n + 2 - k)\}, \qquad k = 3, \cdots, n,$$

and for $n \leqq 40$ about a half dozen iterations are needed to locate the zeros and extreme points of q_n to 16 place accuracy when these initial estimates are used. Finally, we note that the leading coefficient, a_n, of q_n (which corresponds to the leading coefficient 2^{1-n} for the ordinary nth order Chebyshev polynomial) appears to decay with n in such a manner that

$$a_n = (\zeta_{n1} \cdots \zeta_{nn})^{-1} \approx \{\alpha[\alpha + 1/2] \cdots [\alpha + (n-1)/2]\}^{-1},$$

$$\alpha = .276, n = 1, 2, \cdots,$$

with this approximation being good to within about two percent for $n \leqq 40$.

TABLE 1

Zeros and Extremal Points for q_n

n	ζ_{nk}	τ_{nk}	n	ζ_{nk}	τ_{nk}
1	.27846	0.00000	7	0.04364	0.00000
		1.27846		0.39600	0.17509
				1.11914	0.70924
2	0.14728	0.00000		2.25574	1.63180
	1.47277	0.61035		3.88966	3.00276
		3.00971		6.19599	4.94116
				9.65118	7.72085
3	0.09996	0.00000			12.37043
	0.94116	0.40635			
	2.94440	1.75198	8	0.03824	0.00000
		4.82719		0.34636	0.15333
				0.97479	0.61924
4	0.07561	0.00000		1.95077	1.41689
	0.69785	0.30523		3.32411	2.58340
	2.05438	1.27074		5.18551	4.18557
	4.53706	3.10443		7.71882	6.34983
		6.68449		11.41884	9.36171
					14.28748
5	0.06078	0.00000			
	0.55591	0.24456	9	0.03403	0.00000
	1.59954	1.00310		0.30782	0.13638
	3.33784	2.36634		0.86390	0.54968
	6.19974	4.57439		1.72079	1.25311
		8.56540		2.91087	2.27144
				4.48859	3.64662
6	0.05080	0.00000		6.54969	5.45012
	0.46240	0.20406		9.28522	7.81416
	1.31541	0.83046		13.20644	11.03436
	2.68315	1.92781			16.21148
	4.72922	3.60468			
	7.90880	6.12060	10	0.03066	0.00000
		10.46217		0.27702	0.12281
				0.77592	0.49426
				1.54056	1.12387
				2.59332	2.02907
				3.96994	3.23820
				5.72826	4.79650
				7.96803	6.77906
				10.88659	9.32290
				15.01021	12.73268
					18.14115

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