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Multi-Dimensional Extensions of the Chebyshev Polynomials

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Abstract. Two families of polynomials are introduced which satisfy multi-dimensional (or multi-indiced) recursion relationships. These polynomials are developed from the Chebyshev polynomials. Also two additional polynomials are presented which satisfy a special two-dimensional recursion relationship.

I. Introduction. The Chebyshev polynomials belong to the set of ultraspherical or Gegenbauer polynomials and are related to the hypergeometric functions [1]. These polynomials have proven useful in such areas as lattice dynamics [2], numerical analysis [1], and differential equations [1], [3].

The Chebyshev polynomials appear in the literature in various forms, so the following relationships define the forms of the polynomials which will be employed herein [1]:

$$\frac{1-x^2}{1-2\alpha x+x^2} = T(0;\alpha) + 2\sum_{n=1}^{\infty} T(n;\alpha)x^n, \tag{1.1}$$

$$\frac{1}{1-2\alpha x+x^2} = \sum_{n=0}^{\infty} U(n;\alpha)x^n, \tag{1.2}$$

where $T(n;\alpha)$ and $U(n;\alpha)$ are the Chebyshev polynomials of the first and second kind, respectively, and $T(0;\alpha) \equiv 1$.

The terms $(1-x^2)/(1-2\alpha x+x^2)$ and $1/(1-2\alpha x+x^2)$ are the generating functions for the Chebyshev polynomials of the first and second kind, respectively, where the expressions (1.1) and (1.2) are valid, provided $|x| < \min|\alpha \pm (\alpha^2-1)^{1/2}|$.

The expressions for $T(n;\alpha)$ and $U(n;\alpha)$ are

$$T(n;\alpha) = \frac{n}{2}\sum_{m=0}^{[n/2]} \frac{(-)^m(n-m-1)!}{m!\,(n-2m)!}(2\alpha)^{n-2m}, \tag{1.3}$$

$$U(n;\alpha) = \sum_{m=0}^{[n/2]} \frac{(-)^m(n-m)!}{m!\,(n-2m)!}(2\alpha)^{n-2m}. \tag{1.4}$$

Let $I(n;\alpha)$ represent either $T(n;\alpha)$ or $U(n;\alpha)$; then $I(n;\alpha)$ satisfies the recursion relationship

$$2\alpha I(n+1;\alpha) - I(n+2;\alpha) - I(n;\alpha) = 0. \tag{1.5}$$

II. Extensions to Two Dimensions. The Chebyshev polynomials can be extended to two dimensions by forming multivariate generating functions produced

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by replacing α by $(\alpha - (y + y^{-1})/2)$ in the original generating functions. Employing the multinomial theorem, we find that

$$T(n; \alpha - (y + y^{-1})/2) = \sum_{r=-n}^{n} T(n; r; \alpha) y^r, \tag{2.1}$$

$$U(n; \alpha - (y + y^{-1})/2) = \sum_{r=-n}^{n} U(n; r; \alpha) y^r, \tag{2.2}$$

where

$$T(n; r; \alpha) = \frac{n}{2} \sum_{m=0}^{[n/2]} \frac{(-)^{m+r}(n - m - 1)!}{m!} \sum_{k=0}^{[\beta]} \frac{K(2\alpha)^q H(q)}{q!}, \tag{2.3}$$

$$U(n; r; \alpha) = \sum_{m=0}^{[n/2]} \frac{(-)^{m+r}(n - m)!}{m!} \sum_{k=0}^{[\beta]} \frac{K(2\alpha)^q H(q)}{q!} \tag{2.4}$$

subject to the relations

$$\beta = (n - |r| - 2m)/2,$$

$$K = 1/k!\,(k + |r|)!,$$

$$q = n - |r| - 2m - 2k,$$

and where $H(q)$ is the Heaviside step function,

$$H(q) = \begin{Bmatrix} 0 & \text{if } q < 0 \\ 1 & \text{if } q \geqq 0 \end{Bmatrix}.$$

$I(n; r; \alpha)$ satisfies the recursion relationship

$$\begin{aligned} 2\alpha I(n + 1; r + 1; \alpha) - I(n + 2; r + 1; \alpha) - I(n; r + 1; \alpha) \\ - I(n + 1; r + 2; \alpha) - I(n + 1; r; \alpha) = 0, \end{aligned} \tag{2.5}$$

where $I(n; r; \alpha)$ represents either $T(n; r; \alpha)$ or $U(n; r; \alpha)$.

Several of the $U(n; r; \alpha)$ polynomials are displayed in Table I.

TABLE I

$U(n; r; \alpha)$ Polynomials

	r					
n	0	1	2	3	4	5
0	1	0	0	0	0	0
1	2α	-1	0	0	0	0
2	$4\alpha^2 + 1$	-4α	1	0	0	0
3	$8\alpha^3 + 8\alpha$	$-12\alpha^2 - 1$	6α	-1	0	0
4	$16\alpha^4 + 36\alpha^2 + 1$	$-32\alpha^3 - 12\alpha$	$24\alpha^2 + 1$	-8α	1	0
5	$32\alpha^5 + 128\alpha^3 + 18\alpha$	$-80\alpha^4 - 72\alpha^2 - 1$	$80\alpha^3 + 16\alpha$	$-40\alpha^2 - 1$	10α	-1

III. Extensions to $N+1$ Dimensions. The generalization to $N+1$ dimensions is straightforward with the replacement of α by

$$\left(\alpha - \frac{y_1 + y_1^{-1} + y_2 + y_2^{-1} + \cdots + y_N + y_N^{-1}}{2}\right)$$

in the generating functions for the original Chebyshev polynomials.

T and U are given by

$$T(n; r_1, r_2, \cdots, r_N; \alpha) = \frac{n}{2} \sum_{m=0}^{[n/2]} \frac{(-)^{m+\gamma}(n-m-1)!}{m!} \sum_{k_1=0}^{[\beta_1]} K_1 \sum_{k_2=0}^{[\beta_2]} K_2 \cdots \sum_{k_N=0}^{[\beta_N]} \frac{K_N (2\alpha)^q H(q)}{q!}, \tag{3.1}$$

$$U(n; r_1, r_2, \cdots, r_N; \alpha) = \sum_{m=0}^{[n/2]} \frac{(-)^{m+\gamma}(n-m)!}{m!} \sum_{k_1=0}^{[\beta_1]} K_1 \sum_{k_2=0}^{[\beta_2]} K_2 \cdots \sum_{K_N=0}^{[\beta_N]} \frac{K_N (2\alpha)^q H(q)}{q!} \tag{3.2}$$

with

$$\gamma = r_1 + r_2 + \cdots + r_N,$$

$$\beta_p = (n - |r_1| - |r_2| - \cdots - |r_p| - 2m - 2k_1 - 2k_2 - \cdots - 2k_{p-1})/2$$

for $p = 1, 2, 3, \cdots, N$, if we define $k_0 = 0$,

$$K_p = 1/k_p!\,(k_p + |r_p|)!, \qquad p = 1, 2, 3, \cdots, N,$$

$$q = 2(\beta_N - k_N).$$

With mathematical induction, we find that $I(n; r_1, r_2, \cdots, r_N; \alpha)$ satisfies the recursion relationship

$$\sum_{k=0}^{N} \sum_{M_k=-1}^{1} C(M_k, N) I(t_k; S_{1,k}, S_{2,k}, \cdots, S_{N,k}; \alpha) = 0,$$

where

$$C(M_k, N) = \begin{Bmatrix} \dfrac{2\alpha}{N+1} & \text{if } M_k = 0 \\ -1 & \text{if } M_k = -1, 1 \end{Bmatrix},$$

$$t_k = n + 1 + M_0 \delta_{k,0},$$

$$S_{a,k} = r_a + 1 + M_a \delta_{k,a},$$

$$\delta_{k,a} = \begin{Bmatrix} 1 & \text{if } k = a \\ 0 & \text{if } k \neq a \end{Bmatrix},$$

and $I(t_k; S_{1,k}, S_{2,k}, \cdots, S_{N,k}; \alpha)$ represents either $T(t_k; S_{1,k}, S_{2,k}, \cdots, S_{N,k}; \alpha)$ or $U(t_k; S_{1,k}, S_{2,k}, \cdots, S_{N,k}; \alpha)$.

IV. Special Two-Dimensional Polynomials. Sometimes, recursion relationships arise which are similar to Eq. (2.5) but differing in the coefficients of the I's.

Consider the recursion relationship

$$\begin{aligned}
&2\alpha I(n+1; r+1; \beta, \gamma; \alpha) - \beta I(n+2; r+1; \beta, \gamma; \alpha) \\
&\qquad - \beta I(n; r+1; \beta, \gamma; \alpha) - \gamma I(n+1; r+1; \beta, \gamma; \alpha) \\
&\qquad - \gamma I(n+1; r; \beta, \gamma; \alpha) = 0.
\end{aligned} \tag{4.1}$$

An extension of the Chebyshev polynomials allows for the determination of the polynomials which satisfy Eq. (4.1).

Replacing α by

$$\left[\frac{\alpha}{\beta} - \frac{\gamma}{2\beta}(y + y^{-1})\right]$$

in the generating functions produces the polynomials

$$T(n; r; \beta, \gamma; \alpha) = \frac{n}{2} \sum_{m=0}^{[n/2]} \frac{(-)^{m+r}(n-m-1)!}{m!} \left(\frac{\gamma}{\beta}\right)^{n-2m} \sum_{k=0}^{[\beta]} \frac{K(2\alpha/\gamma)^q H(q)}{q!}, \tag{4.2}$$

$$U(n; r; \beta, \gamma; \alpha) = \sum_{m=0}^{[n/2]} \frac{(-)^{m+r}(n-m)!}{m!} \left(\frac{\gamma}{\beta}\right)^{n-2m} \sum_{k=0}^{[\beta]} \frac{K(2\alpha/\gamma)^q H(q)}{q!}, \tag{4.3}$$

where β, K, q, and H are the same as for Eq. (2.4).

The T and U polynomials of Eqs. (4.2) and (4.3) satisfy Eq. (4.1).

V. Comments. A solution to Eq. (1.5), where I does not necessarily represent T or U, can be written in terms of the Chebyshev polynomials. It appears that solutions to the higher-order recursion relationships should consist of combinations of the extended Chebyshev polynomials.

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