Integer Vectors with Interprimed Components*

By Harold N. Shapiro

Abstract. Vectors are considered whose components are positive integers. Such a vector is called interprimed if the components all contain exactly the same distinct prime factors. A method is provided for estimating the number of such vectors, all of whose components are less than a given bound. These estimates resolve a conjecture of Erdös and Motzkin.

1. Introduction. In [1] Erdös and Motzkin raised the question of counting the number of pairs of integers \((a, b), 1 \leq a \leq b \leq x\), such that \(a\) and \(b\) have the same set of distinct prime factors. It is proposed that one show that this number is asymptotic to \(cx\), for some constant \(c\). A solution was proposed [2] which contains an error. Applying different methods, we will provide a solution to Erdös' problem as well as more general ones.

More precisely, a vector \((a_1, \ldots, a_m)\) with positive integral components will be called *interprimed* iff the \(a_i\) all have the same set of distinct prime factors. Letting \(F_m(x)\) equal the number of such vectors with \(1 \leq a_1 \leq a_2 \leq \cdots \leq a_m \leq x\), it can be shown by elementary methods that

\[
F_m(x) = c_m x + O(x^{m/(m+1) + \epsilon}),
\]

for any \(\epsilon > 0\), \(c_m\) a constant depending on \(m\). The case \(m = 2\) is that of Erdös' problem. The details of the proof will be given for the case \(m = 2\). Apart from a certain amount of notational complexity the method carries over to the general case.

Though the method is not pursued here, it should be noted that these problems can also be treated by analytic methods. For example, for the case \(m = 2\) one considers the function \(\Phi(z, w)\) of the two complex variables \(z, w\) defined by

\[
\Phi(z, w) = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} a_{rs}/r^{z}s^{w},
\]

where \(a_{rs} = 1\) if \(r\) and \(s\) have the same distinct prime factors, and \(a_{rs} = 0\) otherwise. The series in (1.2) defines \(\Phi(z, w)\) in the region \(\Re z > 1, \Re w > 1\). It is continuable analytically into the domain \(\Re(z + 2w) > 1, \Re(2z + w) > 1\), by the identity

\[
\Phi(z, w) = G(z, w)\zeta(z + w),
\]

where \(\zeta(s)\) is the Reimann zeta function and \(G(z, w)\) is given by

\[
G(z, w) = \prod_p \left( 1 + \frac{1}{(p^z - 1)(p^w - 1)} \right) \left( 1 + \frac{1}{p^{z+w}} \right).
\]

Received March 23, 1972.


* Results obtained at the Courant Institute of Mathematical Sciences, New York University, with the Office of Naval Research Contract N00014-67-A-0467-0014. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Copyright © 1973, American Mathematical Society
Then $\sum_{r \leq x} \sum_{s \leq x} a_{rs}$ may be approximated by

\begin{equation}
\frac{-1}{4\pi^2} \int_{-iT_1}^{iT_1} \int_{-iT_2}^{iT_2} \Phi(z, w) \frac{z^w}{zw} \, dz \, dw
\end{equation}

where $c = 1 + (1/\log x)$, and $T_1$ and $T_2$ are appropriate functions of $x$. The desired estimation of $\sum_{r \leq x} \sum_{s \leq x} a_{rs}$ results by deforming the contour of integration in (1.3) to where $c = \frac{3}{4} + \epsilon$.

The author is grateful to Professor J. Barlaz for bringing this problem to his attention.

2. Notations. The notations used throughout this note are listed below:

1. $\mu(d) =$ the Möbius function,
2. $(u, v) =$ the greatest common divisor of $u$ and $v$,
3. $\lcm(u, v) =$ the least common multiple of $u$ and $v$,
4. $\text{sq}(m) =$ the squarefree part of $m$, $\text{sq}(1) = 1$ and for $m > 1$,
   $$\text{sq}(m) = \prod_{p|m} p$$
   is the product of the distinct primes dividing $m$. More generally, for any integer $\tau \geq 1$, define
   $$\text{sq}(m, \tau) = \prod_{p|m, \tau \nmid p} p,$$
   that is, the product of the distinct primes which divide $m$ but not $\tau$. Clearly, $\text{sq}(m, 1) = \text{sq}(m)$.
5. $p$ will always be used as a generic symbol for a prime, and $q$ to denote a squarefree integer.
6. $Q(z, A) =$ the number of squarefree integers less than or equal $z$ which are divisible by $A$. Clearly, if $A$ is not squarefree, $Q(z, A) = 0$.
7. $\nu(A) =$ the number of distinct prime factors of $A$.

3. Preliminary Estimates. Various estimates which are needed are cumulated in the following sequence of lemmas.

**Lemma 3.1.** For any fixed integer $A \geq 1$,

$$\sum_{t \leq z; (t, A) = 1} 1 = \prod_{p/A} (1 - 1/p)z + O(2^{\nu(A)})$$

where the $O(2^{\nu(A)})$ is uniform in $A$.

**Proof.** We have

$$\sum_{t \leq z; (t, A) = 1} 1 = \sum_{t \leq z} \sum_{d/(t, A)} \mu(d)$$

$$= \sum_{d/A} \mu(d) \sum_{t \leq z; t \equiv 0 \pmod{d}} 1$$

$$= \sum_{d/A} \mu(d)z/d + O(1)$$

$$= \prod_{p/A} (1 - 1/p)z + O(2^{\nu(A)}).$$
Lemma 3.2. For a fixed squarefree integer $A \geq 1$,

$$Q(z, A) = \frac{z}{A} \sum_d \frac{\mu(d)}{d^2} (d, A) + O(2^{\nu(A)} (z/A)^{1/2})$$

where the $O(2^{\nu(A)} (z/A)^{1/2})$ is uniform in $A$.

Proof.

$$Q(z, A) = \sum_{t \leq \sqrt{z/A}} \mu(d) \sum_{d | t} \frac{1}{d^2}$$

It follows from Lemma 3.1 that the inner sum equals

$$\prod_{p | A} \left(1 - \frac{1}{p}\right) \frac{z}{Ad^2} + O(2^{\nu(A)}),$$

where the error estimate is uniform in $A$. Using this we get

$$Q(z, A) = \sum_{d \leq (z/A)^{1/2}; (d, A) = 1} \mu(d) \left(\prod_{p | A} \left(1 - \frac{1}{p}\right) \frac{z}{Ad^2} + O(2^{\nu(A)})\right)$$

$$= \prod_{p | A} \left(1 - \frac{1}{p}\right) \frac{z}{A} \sum_{d \leq (z/A)^{1/2}; (d, A) = 1} \frac{\mu(d)}{d^2} + O(2^{\nu(A)} (z/A)^{1/2}).$$

Since

$$\sum_{d > (z/A)^{1/2}; (d, A) = 1} \frac{\mu(d)}{d^2} = O((A/z)^{1/2}),$$

this in turn yields

$$Q(z, A) = \prod_{p | A} \left(1 - \frac{1}{p}\right) \frac{z}{A} \sum_{d | A} \frac{\mu(d)}{d^2} + O(2^{\nu(A)} (z/A)^{1/2}).$$

Finally, noting that

$$\prod_{p | A} \left(1 - \frac{1}{p}\right) \sum_{d | A} \frac{\mu(d)}{d^2} = \prod_{p | A} \left(1 - \frac{1}{p}\right) \prod_{\nu(A)} \left(1 - \frac{1}{p^2}\right)$$

$$= \prod_p \left(1 - \frac{(p, A)}{p^2}\right)$$

$$= \sum_d \frac{\mu(d)}{d^2} (d, A),$$

(3.1) is an immediate consequence of (3.2).

Lemma 3.3. Let $\gamma$ and $\tau$ be given positive squarefree integers such that $(\gamma, \tau) = 1$. Then, for any $\epsilon > 0$,

$$\sum_{1 \leq m \leq z; \text{sq}(m, \tau) = 0 \text{ (mod } \gamma)} (\text{sq}(m, \tau))^{-1} = O\left(\frac{z^\epsilon}{\gamma^{1+\epsilon}} \prod_{p \mid \gamma} \frac{1}{1 - 1/p^\epsilon}\right)$$

uniformly in $\gamma$ and $\tau$. (Note that the $O$ does depend on $\epsilon$.)

Proof. We have
\[ \sum_{1 \leq m \leq x, \text{sq}(m, r) = 0 \pmod{\gamma}} (\text{sq}(m, \tau))^{-1} = \sum_{d \mid (q, r)} \frac{1}{d} \sum_{1 \leq m \leq x, \text{sq}(m, r) = 0 \pmod{\gamma}} 1 \]

\[ \leq \sum_{d \mid (q, r)} \frac{1}{d} \sum_{1 \leq m \leq x, \text{sq}(m, r) = 0 \pmod{\gamma}} \left( \frac{z}{m} \right)^{\gamma} \]

\[ \leq z^{\gamma^{\xi}} \sum_{d \mid (q, r)} \frac{1}{d^{\gamma^{\xi}}} \prod_{p \mid q} \frac{1}{1 - p^{\gamma^{\xi}}} \prod_{p \mid r} \frac{1}{1 - 1/p^{\gamma^{\xi}}} \]

\[ \leq O \left( \frac{z^{\gamma^{\xi}}}{\gamma^{\gamma^{\xi}}} \prod_{p \mid q} \frac{1}{1 - 1/p^{\gamma^{\xi}}} \right) \]

as asserted in the lemma.

The above lemma enables us to obtain

**Lemma 3.4.** Let \( B \) and \( r \) be squarefree integers such that \((B, \tau) = 1\). For any given \( \varepsilon_2 > 0 \), there exists a \( c = c(\varepsilon_2), c > 1 \), such that

\[ \sum_{1 \leq m \leq x} \{ B, \text{sq}(m, \tau) \}^{-1} = \mathcal{O} \left( \frac{c^{r(B)}}{B} \prod_{p \mid r} \frac{1}{1 - 1/p^{r(B)}} z^{\gamma^{\xi}} \right) \]

uniformly in \( B \) and \( r \).

**Proof.** We have

\[ \sum_{1 \leq m \leq x} \{ B, \text{sq}(m, \tau) \}^{-1} = \frac{1}{B} \sum_{1 \leq m \leq x} \frac{\{ B, \text{sq}(m, \tau) \}}{\text{sq}(m, \tau)} \]

\[ \leq \frac{1}{B} \sum_{\gamma/B} \gamma \sum_{1 \leq m \leq x, \text{sq}(m, \tau) = 0 \pmod{\gamma}} \frac{1}{\text{sq}(m, \tau)}. \]

Noting that \( \gamma \) will be squarefree and \((\gamma, \tau) = 1\), we apply Lemma 3.3 to the inner sum, so that the above is \( O \) of

\[ \frac{z^{\gamma^{\xi}}}{B} \sum_{\gamma/B} \frac{1}{\gamma^{\gamma^{\xi}}} \prod_{p \mid q} \frac{1}{1 - 1/p^{\gamma^{\xi}}} \leq \frac{z^{\gamma^{\xi}}}{B} \sum_{\gamma/B} \frac{1}{p^{\gamma^{\xi}}} - \prod_{p \mid r} \frac{1}{1 - 1/p^{\gamma^{\xi}}} \]

\[ \leq \frac{z^{\gamma^{\xi}}}{B} \sum_{\gamma/B} c^{r(B)} \frac{1}{p^{\gamma^{\xi}}} - \prod_{p \mid r} \frac{1}{1 - 1/p^{\gamma^{\xi}}} \]

\[ \leq \frac{z^{\gamma^{\xi}}}{B} (c_1 + 1)^{r(B)} \prod_{p \mid r} \frac{1}{1 - 1/p^{\gamma^{\xi}}} \]

(where \( c_1 = (2^{r(B)} - 1)^{-1} \)), and taking \( c = c_1 + 1 \) yields (3.4).

**Lemma 3.5.** Let \( B \) and \( r \) be squarefree integers. For any given \( \varepsilon_3 > 0 \) there exists a constant \( c = c(\varepsilon_3) \) such that
(3.5) \[ \sum_{1 \leq m \leq z : (B, sq(m)) = 0 (\text{mod } \tau)} \{ B, sq(m) \}^{-1} = O\left( \frac{z^{e}}{B^{r}} c^{r(B)} \prod_{p^{r} / \tau} \frac{1}{1 - 1/p^{r}} \right) \]

uniformly in \( B \) and \( \tau \), where \( \tilde{B} = B(B, \tau)^{-1} \).

**Proof.** We have

\[ \sum_{1 \leq m \leq z : (B, sq(m)) = 0 (\text{mod } \tau)} \{ B, sq(m) \}^{-1} \leq \frac{1}{\tau} \sum_{1 \leq m \leq z} \{ \tilde{B}, sq(m, \tau) \}^{-1}. \]

Since \( B \) is squarefree, \( (\tilde{B}, \tau) = 1 \), and applying Lemma 3.4 to the above yields (3.5).

Note that if \( B = sq(l) \) for some integer \( l \), then

\[ \tilde{B} = B(B, \tau)^{-1} = sq(l, \tau) \]

and Lemma 3.5 yields

\[ \sum_{1 \leq m \leq z : (sq(l), sq(m)) = 0 (\text{mod } \tau)} \{ sq(l), sq(m) \}^{-1} \]

(3.6)

\[ = O\left( \frac{z^{e}}{\tau sq(l, \tau)} c^{r(sq(l, \tau))} \prod_{p^{r} / \tau} \frac{1}{1 - 1/p^{r}} \right). \]

Setting \( z = l = m_{2}, m = m_{1}, \tau = 1 \) in (3.6) and summing over all \( m_{2} \leq z \) we obtain

\[ \sum_{1 \leq m_{1}, m_{2} \leq z} \{ sq(m_{1}), sq(m_{2}) \}^{-1} = O\left( z^{e} \sum_{m_{2} \leq z} \frac{c^{r(sq(m_{2}))}}{sq(m_{2})} \right). \]

But it is well known [3] that, for every \( \epsilon > 0 \), \( c^{r(1)} = O(1) \) so that the above is

\[ O\left( z^{e + \epsilon} \sum_{m_{2} \leq z} \frac{1}{sq(m_{2})} \right). \]

Finally, from (3.3) with \( \tau = \gamma = 1, \sum_{m_{2} \leq z} 1/sq(m_{2}) = O(z^{e}) \) so that

\[ (3.7) \sum_{1 \leq m_{1}, m_{2} \leq z} \{ sq(m_{1}), sq(m_{2}) \}^{-1} = O(z^{e}), \]

where \( \epsilon_{5} = \epsilon_{1} + \epsilon_{2} + \epsilon \).

4. The Main Lemma. The results of the previous section are next applied to prove

**Lemma 4.1.** Let \( V \geq U \) be given positive numbers, and let \( S(U, V, d) \) denote the sum

\[ (4.1) \sum_{1 \leq m_{1}, m_{2} \leq V : m_{2} > U} \frac{(d, [sq(m_{1}), sq(m_{2})])}{m_{2}[sq(m_{1}), sq(m_{2})]} \]

Then, for any given \( \epsilon_{4} > 0 \),

\[ (4.2) \sum_{d} \frac{\mu(d)}{d^{2}} S(U, V, d) = O(U^{-1 + \epsilon_{4}}) \]

uniformly in \( V \).

**Proof.** We have
and we apply (3.6) to the inner sum on the right, with \( z = l = m_2 \). This yields that \( S(U, V, d) \) is \( O \) of

\[
\sum_{U < m_2 \leq V} \frac{1}{m_2^{1-\epsilon}} \sum_{\tau / d} c_{\tau}(q(m_2, \tau)) \prod_{p / \tau} \frac{1}{1 - 1/p^{\tau}}
\]

\( \leq \sum_{\tau / d} \prod_{p / \tau} \frac{1}{1 - 1/p^{\tau}} \sum \frac{c_{\tau}(q(m_2, \tau))}{m_2} \]

\( = \sum_{\tau / d} \prod_{p / \tau} \frac{1}{1 - 1/p^{\tau}} \sum \frac{c_{\tau}(q)}{q} \sum \frac{1}{m_2^{1-\epsilon}}. \)

But the inner sum

\[
\sum_{m_2 > U; q(m_2, \tau) = 0} \frac{1}{m_2^{1-\epsilon}} = \frac{1}{q^{1-\epsilon}} \sum_{m_2 > U; q(m_2, \tau) = 0} \frac{1}{m_2^{1-\epsilon}}
\]

\( \leq \frac{1}{q^{1-\epsilon}} \sum_{m_2 > U; q(m_2, \tau) = 0} \frac{1}{m_2^{1-\epsilon}} \) \( \left( \frac{\hat{m}_2 q}{U} \right)^{1-2\epsilon} \)

(we assume \( \epsilon_3 < \frac{1}{2} \)), and this in turn is

\[
\leq \frac{1}{q^{1-\epsilon}} \prod_{p \mid q} \frac{1}{1 - 1/p^{\tau}} \prod_{p / \tau} \frac{1}{1 - 1/p^{\tau}}.
\]

Inserting this in (4.3), we have that \( S(U, V, d) \) is \( O \) of

\[
\sum_{\tau / d} \prod_{p / \tau} \left( \frac{1}{1 - 1/p^{\tau}} \right)^2 \sum_{(q, \tau) = 1} \frac{c_{\tau}(q)}{q^{1+\epsilon}} U^{-1+2\epsilon}. \]

From this it follows that

\[
\sum_{d \mid \mu(d)} \frac{\mu(d)}{d^2} S(U, V, d) = O\left( U^{-1+2\epsilon} \sum_{d \mid \mu(d)} \frac{\mu^2(d)}{d^2} \prod_{p / \tau} \left( 1 - \frac{1}{p^{\tau}} \right)^2 \sum_{(q, \tau) = 1} \frac{c_{\tau}(q)}{q^{1+\epsilon}} \right).
\]

Noting that

\[
\sum_{(q, \tau) = 1} \frac{c_{\tau}(q)}{q^{1+\epsilon}} = O\left( \prod_{p} \left( 1 + \frac{c_2}{p^{1+\epsilon}} \right)^{-1} \right),
\]

this in turn is

\[
O\left( U^{-1+2\epsilon} \prod_{p} \left( 1 + \frac{1}{p^{\tau}} \left( 1 + \left( 1 - \frac{1}{p^{\tau}} \right)^{-2} \right) \left( 1 + \frac{c_2}{p^{1+\epsilon}} \right)^{-1} \right) \right) = O(U^{-1+2\epsilon}).
\]

5. Elementary Proof of (3.1). We will give the details of the proof for the case \( m = 2 \). The general case is completely analogous.

If we consider the integers \( \leq x \) such that the distinct primes which divide them are precisely those which divide the squarefree integer \( q \), these are precisely the integers of the form \( qm \leq x \) such that \( sq(m) \) divides \( q \). Thus we have
We next split the summation so that

\[ F_2(x) = \sum_{1 \leq m_1, m_2 \leq \chi} Q\left(\frac{x}{m_2}, \{\text{sq}(m_1), \text{sq}(m_2)\}\right) \]

or

\[ F_2(x) = \sum_{1 \leq m_1, m_2 \leq \chi} Q\left(\frac{x}{m_2}, \{\text{sq}(m_1), \text{sq}(m_2)\}\right) \]

We note first that since \(2^{2^{t+1}} = O(t^t)\) the \(O\) term is less than

\[ \frac{1}{2^{n+1}} \]

and using (3.7) this is \(O(x^{2/3+\epsilon})\). Thus

\[ S_1 = \chi + O(x^{2/3+\epsilon}) \]

and using (3.7) this is \(O(x^{2/3+\epsilon})\). Thus

\[ S_1 = \chi + O(x^{2/3+\epsilon}) \]

Thus (5.4) becomes

\[ S_1 = cx + O(x^{2/3+\epsilon}) \]

where \(c\) is a constant given by

\[ c = \sum_{d} \frac{\mu(d)}{d^2} \sum_{1 \leq m_1, m_2 \leq \chi/2} \left(\frac{d}{m_2}\right) \left\{\text{sq}(m_1), \text{sq}(m_2)\right\} \]

In fact it is the estimate (5.3) which establishes the convergence of the series.

Turning next to \(S_2\), we have

\[ S_2 \leq \sum_{1 \leq m_1, m_2 \leq \chi} \frac{1}{\{\text{sq}(m_1), \text{sq}(m_2)\}} \]
and from (3.7), this is $O(x^{2/3+\epsilon})$.

Since $F_2(x) = S_1 + S_2$, we have that

$$F_2(x) = cx + O(x^{2/3+\epsilon})$$

as asserted in (1.1).

Courant Institute of Mathematical Sciences
New York University
251 Mercer Street
New York, New York 10012