A New Error Analysis for a Cubic Spline Approximate Solution of a Class of Volterra Integro-Differential Equations

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Abstract. In this paper a third-order numerical method is considered which utilizes a twice continuously differentiable third degree spline to approximate the solution of

\[ x(t) = F(t, x(t), \int_a^t K(t, u, x(u)) \, du), \]

\[ x(a) = x_0, \]

at discrete points in the interval \([a, b]\). The error analysis uses a technique usually associated with linear multistep methods.

I. Introduction. In this paper, consideration is directed to the Volterra integro-differential equation

\[ x(t) = F(t, x(t), \int_a^t K(t, u, x(u)) \, du), \quad a \leq t \leq b, \]

with the initial condition \(x(a) = x_0\). A third-order numerical method is considered which utilizes a twice continuously differentiable third degree spline to approximate the solution \(x\) at discrete points in the interval \([a, b]\).

Other authors, e.g. Hung [5], have applied cubic splines to obtain an approximate solution of a scalar Volterra integro-differential equation. This paper considers the method as applied to vector equations. More important, however, is the error analysis presented herein. This analysis uses a lemma usually associated with linear multistep methods. The utilization of this lemma allows the cubic spline method to be applied to a larger class of equations than considered by Hung with, however, a corresponding reduction in the order of the errors. In particular, Hung requires the solution of (1) be of class \(C^1[a, b]\) while the analysis presented here requires only \(C^0[a, b]\). Accordingly, Hung achieves a discretization error \(O(h^3)\) while this analysis achieves \(O(h^5)\).

II. Notation and Assumptions. Let \(R(F)\) and \(R(K)\) be the regions defined by

\[ R(F) = \{(t, x, y) : a \leq t \leq b; x, y \in E^n\} \]

and

\[ R(K) = \{(t, u, y) : a \leq u \leq t \leq b; y \in E^n\}, \]
where $E^n$ is real Euclidean $n$-space. Moreover, let the $n$th order matrices $F^{(2)}$, $F^{(3)}$, and $K^{(3)}$ be such that the respective elements are given by

$$
F^{(2)}_{i,j} = \frac{\partial F_i}{\partial x_j}, \quad F^{(3)}_{i,j} = \frac{\partial F_i}{\partial y_j}, \quad \text{and} \quad K^{(3)}_{i,j} = \frac{\partial K_i}{\partial y_j}.
$$

Then, the following assumptions are made:

(a) Equation (1) has a unique solution.

(b) $F$ and $K$ are continuous mappings of $R(F)$ and $R(K)$ to $E^n$, respectively.

(c) The matrix elements (2) are continuous and bounded.

Assumption (c) has two important implications. First, there exist constants $F_{12}$, $F_{13}$ and $K_{13}$ such that $\|F^{(2)}\| \leq F_{12}$, $\|F^{(3)}\| \leq F_{13}$ and $\|K^{(3)}\| \leq K_{13}$, where $\| \cdot \|$ will be used interchangeably to denote compatible matrix and vector norms. Secondly, Buck [1, p. 268], for $(t, x, y), (t, \tilde{x}, \tilde{y}) \in R(F)$ there exist $p, \in E^n$, $i = 1, \ldots, n$, such that

$$
F(t, x, y) - F(t, \tilde{x}, \tilde{y}) = F^{(2)}(x - \tilde{x})
$$

where $F^{(2)} = (F^{(2)}_{i,j}(t, p_i, y))$. Similarly, for $(t, x, y), (t, \tilde{x}, \tilde{y}) \in R(F)$ and for $(t, u, v), (t, \tilde{u}, \tilde{v}) \in R(K)$, there are $q_i, r_i \in E^n$, $i = 1, \ldots, n$, such that

$$
F(t, x, y) - F(t, x, \tilde{y}) = F^{(3)}(y - \tilde{y})
$$

and

$$
K(t, u, v) - K(t, u, \tilde{v}) = K^{(3)}(v - \tilde{v})
$$

where $F^{(3)} = (F^{(3)}_{i,j}(t, x, q_i))$ and $K^{(3)} = (K^{(3)}_{i,j}(t, u, r_i))$.

III. The Method. Let $[a, b]$ be divided into $N$ equal subintervals of length $h = (b - a)/N$ with endpoints $t_0, t_1, \ldots, t_N$, called nodes. Let $x_k, \tilde{x}_k, \hat{x}_k$ denote approximations for $x(t_k), \tilde{x}(t_k)$ and $\hat{x}(t_k)$, respectively. The $n$-dimensional cubic spline $S$ on $[a, b]$ is defined as follows: For $t \in [t_k, t_{k+1}]$, $S$ is denoted by $S_k$ and is defined by

$$
S_k(t) = x_k + (t - t_k)\tilde{x}_k + \frac{(t - t_k)^2}{2} \hat{x}_k + \frac{(t - t_k)^3}{3h^2} (\hat{x}_{k+1} - \hat{x}_k - h\hat{x}_k).
$$

Note that $S_k(t_k) = x_k, \dot{S}_k(t_k) = \dot{x}_k, \ddot{S}_k(t_k) = \ddot{x}_k$ and $\dddot{S}_k(t_{k+1}) = \dddot{x}_{k+1}$.

The approximate solution to (1) is obtained by replacing the integral by a numerical quadrature formula and requiring that the resulting equation be satisfied at the nodes. Thus, if the cubic spline $S$ replaces $x$ in this equation, (1) is replaced by

$$
\dddot{S}_k(t_{k+1}) = F(t_{k+1}, S_k(t_{k+1}), h \sum_{i=0}^{k+1} w_i K(t_{k+1}, t_i, S_{i-1}(t_i))
$$

where the weights $w_i$ are bounded and depend on the numerical quadrature formula used and where $S_{-1}(t_0) \equiv x_0$. Then, using $x_k = S_{k-1}(t_k), \dot{x}_k = \dot{S}_{k-1}(t_k), \ddot{x}_k = \ddot{S}_{k-1}(t_k)$ and $\dddot{x}_{k+1} = \dddot{S}_k(t_{k+1})$, (4) becomes

$$
\ddot{x}_{k+1} = H(\dddot{x}_{k+1})
$$

where $H$ is a function of $\dddot{x}_{k+1}$. The numerical results obtained by using this method are presented in the next section.
where

\[ H(\hat{x}_{k+1}) = F(t_{k+1}, x_k + h\hat{x}_k + \frac{h^2}{2} \hat{x}_k + \frac{h}{3} (\hat{x}_{k+1} - \hat{x}_k - h\hat{x}_k), q_{k+1}) \]

with

\[ q_{k+1} = h \sum_{i=0}^{k} w_i K(t_{k+1}, t_i, x_i) \]

\[ + h w_{k+1} K\left(t_{k+1}, t_{k+1}, x_k + h\hat{x}_k + \frac{h^2}{2} \hat{x}_k + \frac{h}{3} (\hat{x}_{k+1} - \hat{x}_k - h\hat{x}_k)\right) \]

All quantities in (5) are known except \( \hat{x}_{k+1} \). Since \( \hat{x}_{k+1} \) determines \( S_k \), the values \( x_{k+1} = S_k(t_{k+1}) \) and \( \hat{x}_{k+1} = \dot{S}_k(t_{k+1}) \) follow. (Although (5) is used to determine \( x_k \), it is convenient to use (4) in the error analysis to follow.)

It follows, in the usual straightforward manner, from assumption (c) that, for \( x \), \( \hat{x} \in E \),

\[ ||H(x) - H(\hat{x})|| \leq \frac{h F^{(2)}(\hat{x}) + h^2 \int w_{k+1} |K^{(3)}(\hat{x})|}{3} ||x - \hat{x}||. \]

Thus, for \( h \) sufficiently small the mapping given by (5) is a contraction. This proves the following theorem.

**Theorem 1.** For \( H \) as defined by (5) and with assumption (c) satisfied, it follows that, for sufficiently small \( h \), \( H \) is a contraction mapping.

Thus, (5) can be used iteratively to determine \( x_i \), \( i = r, \ldots, N \), where \( r \) depends on the starting method used.

**IV. Error Analysis.** Let \( E(t) = x(t) - S(t) \). Then, from (1) and (4), there follows

\[
\begin{align*}
\dot{E}(t_k) &= F\left(t_k, x(t_k), \int_{t_{k-1}}^{t_k} K(t_k, u, x(u)) \, du \right) - F\left(t_k, S_{k-1}(t_k), \int_{t_{k-1}}^{t_k} K(t_k, u, x(u)) \, du \right) \\
&+ F\left(t_k, S_{k-1}(t_k), \int_{t_{k-1}}^{t_k} K(t_k, u, x(u)) \, du \right) \\
&- F\left(t_k, S_{k-1}(t_k), h \sum_{i=0}^{k} w_i K(t_k, t_i, x(t_i)) \right) \\
&+ F\left(t_k, S_{k-1}(t_k), h \sum_{i=0}^{k} w_i K(t_k, t_i, x(t_i)) \right) \\
&- F\left(t_k, S_{k-1}(t_k), h \sum_{i=0}^{k} w_i K(t_k, t_i, S_{k-1}(t_i)) \right).
\end{align*}
\]

Thus, in view of assumption (c),

\[ \dot{E}(t_k) = F^{(2)}(t_k) E(t_k) + \mathcal{O}(h^3) + h F^{(3)}(t_k) \sum_{i=0}^{k} w_i K^{(3)}(t_k, t_i) E(t_i) \]
where $F_{(k)}^{(2)}$ and $F_{(k)}^{(3)}$ indicate the matrices $F^{(2)}$ and $F^{(3)}$ depend on the index $k$ and $K_{(k)}^{(3)}(i)$ indicates the matrix depends on the indices $k$ and $i$. Furthermore, the numerical quadrature formula is assumed to be such that

$$\int_{t_0}^{t_k} K(t, u, x(u)) \, du - h \sum_{i=0}^{k} w_i K(t_k, t_i, x(t_i)) = \mathcal{O}(h^p)$$

where $\mathcal{O}(h^p)$ is a vector with components all $O(h^p)$.

The error analysis development is to obtain an equation involving $E$ and $\ddot{E}$ at the nodes. Then, (6) is used to provide an equation in $E$ only. The error information at the nodes is then used to obtain error bounds at nonnodal points.

To proceed with the error analysis at the nodes, it is assumed the solution $x \in C^4[a, b]$. Then, for $t \in [t_k, t_{k+1}]

(7) \quad E(t) = E(t_k) + (t - t_k) \dot{E}(t_k) + \frac{(t - t_k)^2}{2} \ddot{E}(t_k) + \frac{(t - t_k)^3}{6} \dddot{E}(t_k) + \mathcal{O}(h^3),

(8) \quad \ddot{E}(t) = \ddot{E}(t_k) + (t - t_k) \dddot{E}(t_k) + \frac{(t - t_k)^2}{2} \dddot{E}(t_k) + \mathcal{O}(h^3),

and

(9) \quad \dddot{E}(t) = \dddot{E}(t_k) + (t - t_k) \dddot{E}(t_k) + \mathcal{O}(h^3).

(Since $S^{(4)} = 0$, the error terms involve only the solution $x$ and not the spline $S$.)

Evaluation of (7) and (8) at $t_{k+1}$ and elimination of $\dddot{E}(t_k)$ provides

(10) \quad E(t_{k+1}) = E(t_k) + \frac{2h}{3} \dot{E}(t_k) + \frac{h}{3} \dddot{E}(t_{k+1}) + \frac{h^2}{6} \dddot{E}(t_k) + \mathcal{O}(h^4)

while elimination of $\dddot{E}(t_k)$ provides

(11) \quad \frac{h^3}{12} \dddot{E}(t_k) = E(t_k) - E(t_{k+1}) + \frac{h}{2} \dot{E}(t_k) + \frac{h}{2} \dddot{E}(t_{k+1}) + \mathcal{O}(h^4).

Evaluation of (9) at $t_k$ and substitution of $\dddot{E}(t_k)$ from (11) yields

(12) \quad \dddot{E}(t_{k+1}) = \dddot{E}(t_k) + \frac{12}{h^2} E(t_k) - \frac{12}{h^2} E(t_{k-1}) + \frac{6}{h} \dot{E}(t_k) + \frac{6}{h} \dot{E}(t_{k+1}) + \mathcal{O}(h^3).

Reduction of subscripts by one in (12) and substitution of the resulting expression for $\dddot{E}(t_{k-1})$ into the equation which results from the reduction of the subscripts by one in (10) yields

$$\frac{h^2}{6} \dddot{E}(t_k) = E(t_k) + E(t_{k-1}) + \frac{2h}{3} \dot{E}(t_k) + \frac{h}{3} \dot{E}(t_{k-1}) + \mathcal{O}(h^4),$$

which when substituted in (10) provides

(13) \quad E(t_{k+1}) - E(t_{k-1}) + \frac{h}{3} [\dot{E}(t_{k-1}) + 4 \dot{E}(t_k) + \dot{E}(t_{k+1})] + \mathcal{O}(h^4).

Finally, from (6) and (13), there follows
ERROR ANALYSIS FOR A CUBIC SPLINE SOLUTION

\[
E(t_{k+1}) - E(t_{k-1}) = \frac{h}{3} \left[ F^{(2)}_{(k-1)} E(t_{k-1}) + 4F^{(2)}_{(k)} E(t_k) + F^{(2)}_{(k+1)} E(t_{k+1}) \right]
\]
\[+ \frac{h^2}{3} \left[ F^{(3)}_{(k-1)} \sum_{i=0}^{k-1} w_i K^{(3)}_{(k-1,i)} E(t_i) + 4F^{(3)}_{(k)} \sum_{i=0}^{k} w_i K^{(3)}_{(k,i)} E(t_i) \right]
\[+ F^{(3)}_{(k+1)} \sum_{i=0}^{k+1} w_i K^{(3)}_{(k+1,i)} E(t_i) \]
\[\quad + \tilde{O}(h^{\text{min}(p+1,4)}) \right].
\]

In order to bound the discretization errors at the nodes, the following lemma is used, the proof of which is similar to that for Lemma 5.6 (Henrici, [3, p. 243]) and Linz's [7] lemma.

**Lemma.** Let \( z_m \in E^n, m \geq r \), be the solution of

\[
P_k z_m = x^m + P_{k-1} z_{m+k-1} + \cdots + P_0 z_m
\]

where all \( P_i \) are nth order matrices and the \( \rho_i \) are scalars. Assume the polynomial \( P_k z^k + P_{k-1} z^{k-1} + \cdots + P_0 \) satisfies the Dahlquist stability condition. Thus, if

\[
1/(\rho_k + \rho_{k-1} + \cdots + \rho_0) = \gamma_0 + \gamma_1 t + \cdots,
\]

where \( \rho_k \neq 0 \), then \( \Gamma \equiv \sup_i |\gamma_i| < \infty \) (Henrici [3, p. 242]). Furthermore, assume

\[ ||z_i|| \leq Z, i = 0, 1, \cdots, k + r - 1, \text{ and for all } i, j, ||\beta_{i,j}|| \leq \beta, ||\mu_{i,j}|| \leq \mu, ||\lambda_{i,j}|| \leq \lambda. \]

Then, for sufficiently small \( h \),

\[ ||z_n|| = K^* e^{nhL^*}, \quad n = 0, 1, \cdots, N, \]

where

\[
K^* = [k \Gamma A Z + h b^* \Gamma r Z + N \lambda \Gamma]/[1 - h \Gamma (\beta + b^*)],
\]

\[
L^* = [\beta^* \Gamma + b^* \Gamma]/[1 - h \Gamma (\beta + b^*)],
\]

\[ A = ||\rho_0|| + \cdots + ||\rho_n||, \]

\[ \beta^* = \beta(k + 1) \quad \text{and} \quad \mu^* = \mu(k + 1). \]

Application of the lemma to (14) yields

\[ ||E(t_k)|| \leq K^* e^{L^* t_k} \]

with

\[
K^* = [4Z + 4bwF^{(3)} F^{(3)} r Z + bO(h^{\text{min}(3, p)})]/[1 - h^{(n+1)} F^{(2)} + 4bwF^{(3)} K^{(3)}]
\]

and

\[
L^* = [4F^{(2)} + 4bwF^{(3)} K^{(3)}]/[1 - h^{(n+1)} F^{(2)} + 4bwF^{(3)} K^{(3)}]
\]

where \( |w_i| \leq w, i = 0, \cdots, N \). Z and \( r \) depend on the starting method while \( p \) depends on the numerical integration method used. From \( K^* \) and \( L^* \), it is readily seen that
minimally the starting method should be \( O(h^3) \) and \( p = 3 \). For this case \( ||E(t_0)|| = O(h^3) \). There follows from (6) and (10), respectively, that \( ||\dot{E}(t_0)|| = O(h^3) \) and \( ||\ddot{E}(t_0)|| = O(h) \). This proves the following theorem.

**Theorem 2.** If assumptions (a), (b) and (c) are satisfied, \( x \in C^4[a, b] \) and the starting method and numerical integration method are both \( O(h^3) \), then \( ||E(t_0)|| = O(h^3) \) and \( ||\dot{E}(t_0)|| = O(h) \).

The error analysis at nonnodal points proceeds by setting \( t = t_{k+1} \) in (7) and (8), solving the resulting equations for \( \dot{E}(t_k) \) and \( \ddot{E}(t_k) \) and substituting these equations back into (7) and (8) to obtain

\[
E(t) = \left[ 1 - \frac{3(t - t_k)^2}{h^2} + \frac{2(t - t_k)^3}{h^3} \right] E(t_k) + \left[ \frac{3(t - t_k)^2}{h^2} - \frac{2(t - t_k)^3}{h^3} \right] E(t_{k+1}) \]

and

\[
\dot{E}(t) = \left[ \frac{6(t - t_k)^2}{h^3} - \frac{6(t - t_k)^3}{h^3} \right] E(t_k) + \left[ \frac{6(t - t_k)^2}{h^2} - \frac{6(t - t_k)^3}{h^3} \right] E(t_{k+1}) \]

Hence, for \( t_k \leq t \leq t_{k+1} \),

\[
||E(t)|| \leq ||E(t_k)|| + ||E(t_{k+1})|| + h||\dot{E}(t_k)|| + h||\ddot{E}(t_k)|| + O(h^4)
\]

and

\[
||\dot{E}(t)|| \leq \frac{1}{h} ||E(t_k)|| + \frac{1}{h} ||E(t_{k+1})|| + \frac{1}{h} ||\dot{E}(t_k)|| + \frac{1}{h} ||\ddot{E}(t_k)|| + O(h^3).
\]

Thus, \( ||E(t)|| = O(h^3) \) and \( ||\dot{E}(t)|| = O(h^3) \). From (9) and (11) there follows \( ||\ddot{E}(t)|| = O(h) \). This proves the following theorem.

**Theorem 3.** If the conditions of Theorem 2 are satisfied, then, for \( t \in [a, b] \), \( t \) a nonnodal point, \( ||E(t)|| = O(h^3) \), \( ||\dot{E}(t)|| = O(h^3) \) and \( ||\ddot{E}(t)|| = O(h) \).

**V. A Numerical Example.** The scalar equation considered here is

\[
x(t) = -\frac{3}{16} (t - 1)x(t) + \int_0^t x(u) \, du + \frac{13}{3} (t - 1)^{10/3} + \frac{3}{16},
\]

\( x(0) = -1 \),

which has the solution \( x(t) = (t - 1)^{13/3} \). Gregory's third-order formula,  

\[
\int_{t_0}^{t_k} f = h \left( \sum_{i=0}^{k} \frac{5}{12} f_i + \frac{13}{12} f_0 + f_1 + \cdots + f_{k-2} + \frac{13}{12} f_{k-1} + \frac{5}{12} f_k \right),
\]

where \( f(t) = x(t) \).
is used for numerical quadrature. In order to apply Gregory's formula, the values \( x_0, x_1, x_2 \) and \( x_3 \) are needed. \( x_0 \) is known. To obtain the other starting values, let 
\[ G(t, x(t)) \]
represent the right side of the above equation, i.e., 
\[ x(t) = G(t, x(t)) \]
and let \( G_k = G(t_k, x(t_k)) \). Then

\[
x(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} G(t, x(t)) \, dt
\]

and

\[
x(t_{k+2}) = x(t_k) + \int_{t_k}^{t_{k+2}} G(t, x(t)) \, dt.
\]

Thus, using Simpson's rule,

(15) \[ x_{k+1} = x_k + \frac{h}{6} (G_k + 4G_{k+1/2} + G_{k+1}) \]

and

(16) \[ x_{k+2} = x_k + \frac{h}{3} (G_k + 4G_{k+1} + G_{k+2}) \]

where \( t_{k+1/2} = t_k + \frac{h}{2} \). The quadratic equation through the points \((t_k, x_k), (t_{k+1}, x_{k+1})\) and \((t_{k+2}, x_{k+2})\) evaluated at \( t_{k+1/2} \) provides

(17) \[ x_{k+1/2} = \frac{3}{8} x_k + \frac{3}{4} x_{k+1} - \frac{1}{8} x_{k+2}, \]

and, similarly,

(18) \[ G_{k+1/2} = \frac{3}{8} G_k + \frac{3}{4} G_{k+1} - \frac{1}{8} G_{k+2} . \]

### Table of Errors

| Step Size 2^{-P} | \( |E(1.5)| \) | \( |\dot{E}(1.5)| \) | max \( |E(i, t)| \), \( i = 1, 2, 3 \) (for starting method) |
|------------------|----------------|------------------|------------------------------------------------|
| 2                | \( .865964 \times 10^{-3} \) | \( .170270 \times 10^{-2} \) | \( .636479 \times 10^{-2} \) |
| 3                | \( .915525 \times 10^{-3} \) | \( .101191 \times 10^{-2} \) | \( .423563 \times 10^{-3} \) |
|                  | \( (.108248 \times 10^{-2}) \) | \( (.212838 \times 10^{-3}) \) |                                        |
| 4                | \( .227987 \times 10^{-3} \) | \( .218648 \times 10^{-3} \) | \( .274553 \times 10^{-4} \) |
|                  | \( (.114481 \times 10^{-3}) \) | \( (.126889 \times 10^{-3}) \) |                                        |
| 5                | \( .380606 \times 10^{-4} \) | \( .352258 \times 10^{-4} \) | \( .174974 \times 10^{-5} \) |
|                  | \( (.284984 \times 10^{-4}) \) | \( (.273310 \times 10^{-4}) \) |                                        |
| 6                | \( .545480 \times 10^{-5} \) | \( .498293 \times 10^{-5} \) | \( .110465 \times 10^{-6} \) |
|                  | \( (.475757 \times 10^{-5}) \) | \( (.440322 \times 10^{-5}) \) |                                        |
| 7                | \( .728986 \times 10^{-6} \) | \( .662161 \times 10^{-6} \) | \( .693943 \times 10^{-8} \) |
|                  | \( (.681850 \times 10^{-6}) \) | \( (.622616 \times 10^{-6}) \) |                                        |
| 8                | \( .941891 \times 10^{-7} \) | \( .853280 \times 10^{-7} \) | \( .434832 \times 10^{-9} \) |
|                  | \( (.911232 \times 10^{-7}) \) | \( (.827701 \times 10^{-7}) \) |                                        |
Equations (15), (16), (17) and (18) are used iteratively to find \( x_1 \) and \( x_2 \), then \( x_3 \) and \( x_4 \). The value \( x_4 \) is not a needed starting value but it is obtained simultaneously with \( x_3 \). The method is used until the difference between two consecutive iterates does not exceed \( 10^{-12} \). This starting method is \( O(h^4) \).

Once the starting values are known, the method of Section III is used with Gregory's formula for the numerical integration. Note the method is applied to \( x_i \), not \( \dot{x}_i \). Once \( \dot{x}_i \) is known, then

\[
\dot{x}_i = \ddot{S}_{i-1}(t_i) \quad \text{and} \quad \dot{x}_i = \dddot{S}_{i-1}(t_i).
\]

The above table summarizes the errors corresponding to various step sizes. According to the theory, if the step size is halved the errors in \( E \) and \( \dot{E} \) should be reduced by approximately one-eighth. The numbers in parenthesis are one-eighth the error for the previous step size, i.e., the error predicted by the theory when the step size is halved.

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