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A First Order Method for Differential Equations of Neutral Type

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Abstract. A first order method is presented for solution of the initial-value problem for a differential equation of neutral type with implicit delay in the critical case where the time-lag is zero and the method of stepwise integration does not apply. A convergence theorem is proved, and numerical examples are given.

1. Introduction. In this note, we present a first order method for the numerical solution of the initial-value problem (IVP) for a neutral-type functional-differential equation without previous history:

$$x'(t) = f(t, x(t), x(g(t, x(t))), x'(g(t, x(t)))), \tag{1}$$

$$x(a) = x_0, \qquad x'(a) = z_0, \tag{2}$$

where z_0 is a real root of the algebraic equation

$$z = f(a, x_0, x_0, z). \tag{3}$$

Here, $x(t)$ is a scalar function to be determined on some finite interval $[a, b]$. We shall make the following assumptions regarding f and g:

(H1) f and g are continuous and satisfy uniform Lipschitz conditions of the form

$$|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leqq L\{|x_1 - x_2| + |y_1 - y_2|\} + L_z\,|z_1 - z_2|,$$

$$|g(t, x_1) - g(t, x_2)| \leqq L_g|x_1 - x_2|$$

in their respective domains E and E', where

$$E = \{(t, x, y, z): a \leqq t \leqq b,\ |x - x_0| \leqq c,\ |y - x_0| \leqq c,\ |z| \leqq M\}$$

and E' is the projection of E in the (t, x) space; c, M, L, L_g, L_z are constants, with $L_z < 1$, M is such that $\sup_{(t,x,y,z)\in E} |f(t, x, y, z)| < M$, and $M(b - a) < c$.

(H2) $a \leqq g(t, x) \leqq t$ for $(t, x) \in E'$.

Our hypotheses, together with additional smoothness and growth conditions on f and g, ensure the local existence of a solution of the IVP (1)–(2). Furthermore, $x(t)$ is the only solution having a bounded derivative on $[a, b]$; see [2], [4]. Our result extends a method developed by Feldstein [3] for the equation of retarded type

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$$x'(t) = f(t, x(t), x(g(t)))$$

to the neutral-type equation with implicit delay (1). Other methods for implicit-delay equations are given in [1].

2. The Algorithm $\mathfrak{A}$. Let $y(t) = x(g(t, x(t)))$; $z(t) = x'(g(t, x(t)))$. Let N be a positive integer, and let $h = (b - a)/N$. For each nonnegative integer $n \leqq N$, let $t_n = a + nh$. Let $[s]$ denote the integer part of s. Define the algorithm $\mathfrak{A}$ as follows:

$$f_n = f(t_n, x_n, y_n, z_n), \qquad g_n = g(t_n, x_n), \tag{4}$$

$$q(n) = [(g_n - a)/h], \qquad r(n) = (g_n - a)/h - q(n), \tag{5}$$

$$y_0 = x_0, \qquad y_n = x_{q(n)} + hr(n)f_{q(n)}, \tag{6}$$

$$z_n = f_{q(n)}, \tag{7}$$

$$x_{n+1} = x_n + hf_n. \tag{8}$$

Note that condition (H2) implies $q(n) \leqq n$, thus, the algorithm is well defined. For $n = 0$, $g_0 = a$, $q(0) = 0$, and $r(0) = 0$. Thus, $y_0 = x_0$ and $z_0 = f(a, x_0, x_0, z_0)$. Let u_0, an approximation of the root z_0, be chosen independently of h. It is of interest to note that such an approximation does not destroy the order h convergence of the algorithm. It is of further interest that (6) may be simplified to $y_n = x_{q(n)}$. The error bound established in the convergence theorem for this "simplified" algorithm is larger but still of order h, as noted following the proof of convergence of the algorithm $\mathfrak{A}$. The second numerical example of Section 4 demonstrates both the algorithm $\mathfrak{A}$ and the simplified algorithm.

If $g_n = t_n$ for any n, $1 \leqq n \leqq N$, then $q(n) = n$, $r(n) = 0$, and (7) becomes $z_n = f(t_n, x_n, y_n, z_n)$ which has exactly one root z in the interval $[-M, M]$ under the conditions (H1)–(H2) together with the smoothness and growth conditions mentioned in Section 1. We must in general include a procedure for finding this root, and this in turn will affect the error estimate. As before, such an estimate does not destroy the order h convergence of the algorithm. For simplicity, we do not take this into account, since our aim is to show the convergence of the algorithm $\mathfrak{A}$.

Thus, we shall assume in the convergence proof that (7) will not reduce to $z_n = f(t_n, x_n, y_n, z_n)$, $n \geqq 1$.

3. Convergence.

THEOREM. *Let f and g satisfy* (H1)–(H2) *and suppose, in addition, that there exists a unique solution $x(t)$ of* (1)–(2) *with* $\sup_{[a,b]} |x''(t)| \leqq B$. *Then, for each $t_n \in [a, b]$, $0 < n \leqq N$,*

$$|x_n - x(t_n)| \leqq h\left\{L_z\, |z_0 - u_0|\, e^{s(b-a)} + \frac{B}{2s}\left(\frac{1 + L_z}{1 - L_z}\right)(e^{s(b-a)} - 1)\right\} + O(h^2)$$

where

$$s = L(1 + c_0) + L_z c_1,$$

$$c_0 = 1 + ML_g,$$

$$c_1 = (L(2 + ML_g) + BL_g)/(1 - L_z),$$

u_0 is the approximation to z_0 mentioned above, and x_n is given by algorithm $\mathfrak{A}$.

Proof. Let $e_n = |x_n - x(t_n)|$; $e_n^* = |y_n - y(t_n)|$; $e_n^{**} = |z_n - z(t_n)|$. From (8) and Taylor's formula, we obtain

$$e_{n+1} \leqq e_n + h(L(e_n + e_n^*) + L_z e_n^{**}) + h^2B/2. \tag{9}$$

Equation (5) implies that $g_n = t_{q(n)} + hr(n)$, and hence, in a similar manner, we have (after replacing n by $(n + 1)$)

$$\begin{aligned} e_{n+1}^* \leqq\ & ML_g e_{n+1} + e_{q(n+1)} \\ & + hr(n+1)\{L(e_{q(n+1)} + e_{q(n+1)}^*) + L_z e_{q(n+1)}^{**}\} + h^2r^2(n+1)B/2, \end{aligned} \tag{10}$$

$$e_{n+1}^{**} \leqq BL_g e_{n+1} + L(e_{q(n+1)} + e_{q(n+1)}^*) + L_z e_{q(n+1)}^{**} + hr(n+1)B. \tag{11}$$

We then have two cases to consider:

Case 1. $q(n + 1) = n + 1$ and $r(n + 1) = 0$. Under these conditions, (9) is unchanged:

$$e_{n+1} \leqq e_n(1 + hL) + e_n^* hL + e_n^{**} hL_z + h^2B/2. \tag{9a}$$

(10) becomes

$$e_{n+1}^* \leqq e_{n+1}(1 + ML_g) = e_{n+1}c_0. \tag{10a}$$

And (11) becomes

$$e_{n+1}^{**} \leqq (L + BL_g)e_{n+1} + Le_{n+1}^* + L_z e_{n+1}^{**}$$

or

$$e_{n+1}^{**} \leqq \left(\frac{L + BL_g + L(1 + ML_g)}{1 - L_z}\right)e_{n+1} = e_{n+1}c_1. \tag{11a}$$

Define the partial ordering for vectors: $v_1 = (v_1^1, \cdots, v_1^k) \leqq v_2 = (v_2^1, \cdots, v_2^k)$ if $v_1^i \leqq v_2^i$, $i = 1, \cdots, k$. Then, in vector form, (9a), (10a), and (11a) become

$$\begin{bmatrix} e_{n+1} \\ e_{n+1}^* \\ e_{n+1}^{**} \end{bmatrix} \leqq \begin{bmatrix} 1 + hL & hL & hL_z \\ (1 + hL)c_0 & hLc_0 & hL_zc_0 \\ (1 + hL)c_1 & hLc_1 & hL_zc_1 \end{bmatrix} \begin{bmatrix} e_n \\ e_n^* \\ e_n^{**} \end{bmatrix} + hB \begin{bmatrix} h/2 \\ hc_0/2 \\ hc_1/2 \end{bmatrix}$$

which is of the form $d_{n+1} \leqq A_1 d_n + b_1$.

Case 2. $q(n + 1) \leqq n$ and $0 \leqq r(n + 1) < 1$.

Let

$$\delta_n = \max_{1 \leqq i \leqq n} e_i, \qquad \delta_n^* = \max_{1 \leqq i \leqq n} e_i^*, \qquad \delta_n^{**} = \max_{1 \leqq i \leqq n} e_i^{**}.$$

Then, (9) becomes

$$\delta_{n+1} \leqq \delta_n(1 + hL) + \delta_n^* hL + \delta_n^{**} hL_z + h^2B/2. \tag{9b}$$

And (10) becomes

$$\delta^*_{n+1} \leqq ML_g\delta_{n+1} + \delta_n(1 + hL) + hL\delta^*_n + hL_z\delta^{**}_n + h^2B/2.$$

Using (9b), we have

$$\delta^*_{n+1} \leqq (\delta_n(1 + hL) + \delta^*_n hL + \delta^{**}_n hL_z + h^2B/2)(1 + ML_g)$$

or

$$\text{(10b)} \qquad \delta^*_{n+1} \leqq \delta_n(1 + hL)c_0 + \delta^*_n hLc_0 + \delta^{**}_n hL_z c_0 + h^2c_0B/2.$$

Finally, (11) becomes

$$\delta^{**}_{n+1} \leqq \delta_{n+1}BL_g + \delta_n L + \delta^*_n L + \delta^{**}_n L_z + hB.$$

Further, enlarging δ_n to δ_{n+1} and δ^*_n to δ^*_{n+1} on the right, and using $1 - L_z > 0$, we find

$$\delta^{**}_{n+1} \leqq \delta_{n+1}\left(\frac{L + BL_g}{1 - L_z}\right) + \delta^*_{n+1}\frac{L}{1 - L_z} + \frac{hB}{1 - L_z}.$$

Using (9b) and (10b), we have

$$\delta^{**}_{n+1} \leqq \left(\frac{L + BL_g + Lc_0}{1 - L_z}\right)\left(\delta_n(1 + hL) + \delta^*_n hL + \delta^{**}_n hL_z + \frac{h^2B}{2}\right) + \frac{hB}{1 - L_z}$$

or

$$\text{(11b)} \qquad \delta^{**}_{n+1} \leqq \delta_n(1 + hL)c_1 + \delta^*_n hLc_1 + \delta^{**}_n hL_z c_1 + \frac{hB}{1 - L_z} + \frac{h^2c_1B}{2}.$$

Then, as a vector system, (9b), (10b), and (11b) become

$$\text{(12)} \qquad \begin{bmatrix} \delta_{n+1} \\ \delta^*_{n+1} \\ \delta^{**}_{n+1} \end{bmatrix} \leqq \begin{bmatrix} 1 + hL & hL & hL_z \\ (1 + hL)c_0 & hLc_0 & hL_z c_0 \\ (1 + hL)c_1 & hLc_1 & hL_z c_1 \end{bmatrix} \begin{bmatrix} \delta_n \\ \delta^*_n \\ \delta^{**}_n \end{bmatrix} + hB \begin{bmatrix} h/2 \\ hc_0/2 \\ hc_1/2 + 1/(1 - L_z) \end{bmatrix}$$

which is of the form $d_{n+1} \leqq A_2 d_n + b_2$. Comparing this with the result obtained in Case 1, we find that A_1 and A_2 are identical and that $b_1 \leqq b_2$. Thus, any bound obtained here in Case 2 for d_{n+1} will also bound d_{n+1} in Case 1.

To complete the proof, we shall use the following lemmas [3] which may be verified by induction:

LEMMA 1. *Suppose A is a $k \times k$ real matrix and b is a real k-vector. Let $\{d_n\}$ ($n = 0, 1, \cdots$) satisfy $d_{n+1} \leqq Ad_n + b$. Then*

$$d_{n+1} \leqq A^{n+1}d_0 + \left(\sum_{i=0}^{n} A^i\right)b.$$

LEMMA 2. *Let $p = (p_1, \cdots, p_k)$, $q = (q_1, \cdots, q_k)$. Suppose the $k \times k$ matrix A has the form $A = p^Tq$. Then*

$$A^n = \left(\sum_{i=1}^{k} p_i q_i\right)^{n-1} A.$$

By Lemma 1,

$$d_{n+1} \leqq A_2^{n+1} d_0 + \left(\sum_{i=0}^{n} A_2^i \right) b_2,$$

where

$$d_0 = \begin{bmatrix} e_0 \\ e_0^* \\ e_0^{**} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ |z_0 - u_0| \end{bmatrix}.$$

Then, because

$$A_2 = \begin{bmatrix} 1 \\ c_0 \\ c_1 \end{bmatrix} (1 + hL, hL, hL_z),$$

we can make use of Lemma 2 to obtain

$$A_2^i = (1 + hL + hLc_0 + hL_z c_1)^{i-1} A_2 = (1 + hs)^{i-1} A_2.$$

Two results follow from this: $A_2^{n+1} = (1 + hs)^n A_2 \leqq e^{s(b-a)} A_2$, and

$$\sum_{i=1}^{n} A_2^i = A_2 \sum_{i=1}^{n} (1 + hs)^{i-1} = \frac{((1 + hs)^n - 1)}{hs} A_2 \leqq \frac{1}{hs} (\exp(s(b - a)) - 1) A_2.$$

Finally,

$$\begin{aligned} d_{n+1} &\leqq A_2^{n+1} d_0 + \left(\sum_{i=0}^{n} A_2^i \right) b_2 \\ &\leqq h \left\{ |z_0 - u_0| \, L_z e^{s(b-a)} \begin{bmatrix} 1 \\ c_0 \\ c_1 \end{bmatrix} \right. \\ &\qquad \left. + \frac{B}{2s} \left(hs + \frac{1 + L_z}{1 - L_z} \right) (e^{s(b-a)} - 1) \begin{bmatrix} 1 \\ c_0 \\ c_1 \end{bmatrix} + B \begin{bmatrix} \frac{h}{2} \\ \frac{hc_0}{2} \\ \frac{hc_1}{2} + \frac{1}{1 - L_z} \end{bmatrix} \right\} \end{aligned}$$

which gives

$$e_{n+1} \leqq \delta_{n+1} \leqq h \left\{ |z_0 - u_0| \, L_z e^{s(b-a)} + \frac{B}{2s} \left(hs + \frac{1 + L_z}{1 - L_z} \right) (e^{s(b-a)} - 1) + \frac{hB}{2} \right\}$$

and the theorem follows.

For the simplified algorithm, where (6) is replaced by $y_n = x_{q(n)}$ the following bound is possible:

$$
\begin{aligned}
d_{n+1} \leqq h\Bigg\{ &|z_0 - u_0|\ L_z e^{s(b-a)} \begin{bmatrix} 1 \\ c_0 \\ c_1 \end{bmatrix} \\
&+ \left(\frac{B}{2s}\left(hs + \frac{1+L_z}{1-L_z}\right) + \frac{1}{s}\left(\frac{ML}{1-L_z}\right)\right)(e^{s(b-a)} - 1) \begin{bmatrix} 1 \\ c_0 \\ c_1 \end{bmatrix} \\
&+ B \begin{bmatrix} \dfrac{h}{2} \\ \dfrac{hc_0}{2} \\ \dfrac{hc_1}{2} + \dfrac{1}{1-L_z} \end{bmatrix} + \begin{bmatrix} 0 \\ M \\ \dfrac{ML}{1-L_z} \end{bmatrix} \Bigg\},
\end{aligned} \tag{13}
$$

and hence

$$
\begin{aligned}
e_{n+1} \leqq h\Big\{ &|z_0 - u_0|\ L_z e^{s(b-a)} \\
&+ \left(\frac{B}{2s}\left(hs + \frac{1+L_z}{1-L_z}\right) + \frac{1}{s}\left(\frac{ML}{1-L_z}\right)\right)(e^{s(b-a)} - 1) + \frac{hB}{2}\Big\}.
\end{aligned}
$$

TABLE I. *$x_n(h)$ denotes the value of x_n for step size h.*

t_n	$x(t_n)$	$x_n(2^{-4})$	$x_n(2^{-6})$	$x_n(2^{-8})$	$x_n(2^{-10})$
0	0	0	0	0	0
.0625	.0039	0	.0029	.0034	.0039
.1250	.0158	.0078	.0138	.0153	.0157
.1875	.0360	.0238	.0329	.0352	.0358
.2500	.0653	.0484	.0610	.0642	.0650
.3125	.1048	.0825	.0990	.1032	.1044
.3750	.1562	.1275	.1485	.1541	.1556
.4375	.2224	.1853	.2119	.2196	.2217
.5000	.3078	.2593	.2942	.3043	.3069
.5625	.4206	.3547	.4026	.4159	.4194
.6250	.5771	.4856	.5518	.5705	.5754
.6875	.8185	.6707	.7778	.8080	.8159
.7500	1.3244	.9860	1.2205	1.2968	1.3174

TABLE II. *$x_n^{(1)}(h)$ denotes the value of x_n for step size h by algorithm $\mathfrak{A}$; $x_n^{(2)}(h)$ denotes the value of x_n for step size h by the simplified algorithm.*

t_n	$x(t_n)$	$x_n^{(1)}(2^{-2})$	$x_n^{(2)}(2^{-2})$	$x_n^{(1)}(2^{-4})$	$x_n^{(2)}(2^{-4})$
.25	.2474	.2500	.2500	.2483	.2478
.50	.4794	.4930	.4892	.4838	.4759
.75	.6816	.7180	.6866	.6942	.6739
1.00	.8414	.9228	.8569	.8697	.8273
t_n	$x(t_n)$	$x_n^{(1)}(2^{-8})$	$x_n^{(2)}(2^{-8})$	$x_n^{(1)}(2^{-12})$	$x_n^{(2)}(2^{-12})$
.25	.2474	.2475	.2471	.2474	.2474
.50	.4794	.4797	.4787	.4794	.4794
.75	.6816	.6825	.6802	.6817	.6815
1.00	.8414	.8435	.8390	.8416	.8413

4. Examples. (a) We solve the IVP

$$x'(t) = \frac{-4tx^2(t)}{4 + \log^2 \cos t} + \tan 2t + \tfrac{1}{2} \tan^{-1} z$$

($z_0 = 0$, $x_0 = 0$, $z = x'(g(t, x(t))) \equiv x'(tx^2(t)/(1 + x^2(t)))$) on the interval [0, .75]. The existence and uniqueness of the solution is guaranteed by the results of [2] mentioned earlier. The only solution is $x(t) = -\frac{1}{2} \log \cos 2t$.

The results of the computation by algorithm $\mathfrak{A}$ are given in Table I.

(b) Consider the IVP

$$x'(t) = \cos t(1 + y) + xz - \sin(t(1 + \sin^2 t)),$$

with $y = x(tx^2(t))$, $z = x'(tx^2(t))$, $z_0 = 1$, $x_0 = 0$, on the interval [0, 1]. As in example (a), existence and uniqueness of the solution are guaranteed by the results of [2]. Here, the solution is $x(t) = \sin t$.

The results of the computation by the algorithm $\mathfrak{A}$ and by the simplified algorithm are given in Table II.

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