A First Order Method for Differential Equations of Neutral Type

By R. N. Castleton and L. J. Grimm*

Abstract. A first order method is presented for solution of the initial-value problem for a differential equation of neutral type with implicit delay in the critical case where the time-lag is zero and the method of stepwise integration does not apply. A convergence theorem is proved, and numerical examples are given.

1. Introduction. In this note, we present a first order method for the numerical solution of the initial-value problem (IVP) for a neutral-type functional-differential equation without previous history:

\[
\begin{aligned}
&x'(t) = f(t, x(t), x(g(t, x(t)))); \\
&x(a) = x_0, \quad x'(a) = z_0,
\end{aligned}
\]

where \(z_0\) is a real root of the algebraic equation

\[
z = f(a, x_0, x_0, z).
\]

Here, \(x(t)\) is a scalar function to be determined on some finite interval \([a, b]\). We shall make the following assumptions regarding \(f\) and \(g\):

(H1) \(f\) and \(g\) are continuous and satisfy uniform Lipschitz conditions of the form

\[
|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq L_1 |x_1 - x_2| + L_2 |y_1 - y_2| + L_3 |z_1 - z_2|,
\]

\[
|g(t, x_1) - g(t, x_2)| \leq L_4 |x_1 - x_2|
\]

in their respective domains \(E\) and \(E'\), where

\[
E = \{(t, x, y, z): a \leq t \leq b, |x - x_0| \leq c, |y - x_0| \leq c, |z| \leq M\}
\]

and \(E'\) is the projection of \(E\) in the \((t, x)\) space; \(c, M, L, L_1, L_2, L_3, L_4\) are constants, with \(L_4 < 1, M\) is such that \(\sup_{(t, x, y, z) \in E} |f(t, x, y, z)| < M\), and \(M(b - a) < c\).

(H2) \(a \leq g(t, x) \leq t\) for \((t, x) \in E'\).

Our hypotheses, together with additional smoothness and growth conditions on \(f\) and \(g\), ensure the local existence of a solution of the IVP (1)–(2). Furthermore, \(x(t)\) is the only solution having a bounded derivative on \([a, b]\); see [2], [4]. Our result extends a method developed by Feldstein [3] for the equation of retarded type
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to the neutral-type equation with implicit delay (1). Other methods for implicit-delay equations are given in [1].

2. The Algorithm \( \mathcal{A} \). Let \( y(t) = x(g(t, x(t))); z(t) = x'(g(t, x(t))) \). Let \( N \) be a positive integer, and let \( h = (b - a)/N \). For each nonnegative integer \( n \leq N \), let \( t_n = a + nh \). Let \([s]\) denote the integer part of \( s \). Define the algorithm \( \mathcal{A} \) as follows:

\[
\begin{align*}
(4) & \quad f_n = f(t_n, x_n, y_n, z_n), \quad g_n = g(t_n, x_n), \\
(5) & \quad q(n) = \left[\frac{(g_n - a)}{h}\right], \quad r(n) = \frac{(g_n - a)}{h} - q(n), \\
(6) & \quad y_0 = x_0, \quad y_n = x_{q(n)} + hr(n)f_{q(n)}, \\
(7) & \quad z_n = f_{q(n)}, \\
(8) & \quad x_{n+1} = x_n + hf_n.
\end{align*}
\]

Note that condition (H2) implies \( q(n) \leq n \), thus, the algorithm is well defined. For \( n = 0 \), \( g_0 = a \), \( q(0) = 0 \), and \( r(0) = 0 \). Thus, \( y_0 = x_0 \) and \( z_0 = f(a, x_0, x_0, z_0) \). Let \( u_0 \), an approximation of the root \( z_0 \), be chosen independently of \( h \). It is of interest to note that such an approximation does not destroy the order \( h \) convergence of the algorithm. It is of further interest that (6) may be simplified to \( y_n = x_{q(n)} \). The error bound established in the convergence theorem for this “simplified” algorithm is larger but still of order \( h \), as noted following the proof of convergence of the algorithm \( \mathcal{A} \). The second numerical example of Section 4 demonstrates both the algorithm \( \mathcal{A} \) and the simplified algorithm.

If \( g_n = t_n \) for any \( n \), \( 1 \leq n \leq N \), then \( q(n) = n \), \( r(n) = 0 \), and (7) becomes \( z_n = f(t_n, x_n, y_n, z_n) \) which has exactly one root \( z \) in the interval \([ -M, M] \) under the conditions (H1)-(H2) together with the smoothness and growth conditions mentioned in Section 1. We must in general include a procedure for finding this root, and this in turn will affect the error estimate. As before, such an estimate does not destroy the order \( h \) convergence of the algorithm. For simplicity, we do not take this into account, since our aim is to show the convergence of the algorithm \( \mathcal{A} \).

Thus, we shall assume in the convergence proof that (7) will not reduce to \( z_n = f(t_n, x_n, y_n, z_n), \ n \geq 1 \).

3. Convergence.

THEOREM. Let \( f \) and \( g \) satisfy (H1)-(H2) and suppose, in addition, that there exists a unique solution \( x(t) \) of (1)-(2) with \( \sup_{[a, b]} |x'''(t)| \leq B \). Then, for each \( t_n \in [a, b], 0 < n \leq N \),

\[
|x_n - x(t_n)| \leq h \left( L_s |z_0 - u_0| \ e^{s(h-a)} + \frac{B}{2s} \left( \frac{1 + L_c}{1 - L_c} \right) \left( e^{s(h-a)} - 1 \right) \right) + O(h^2)
\]

where

\[
s = L(1 + c_0) + L_c c_1,
\]
\[ c_0 = 1 + ML_0, \]
\[ c_1 = (L(2 + ML_0) + BL_0)/(1 - L), \]

\( u_0 \) is the approximation to \( z_0 \) mentioned above, and \( x_0 \) is given by algorithm \( A \).

**Proof.** Let \( e_n = |x_n - x(t_n)|; \ e_n^* = |y_n - y(t_n)|; \ e_n^{**} = |z_n - z(t_n)| \). From (8) and Taylor’s formula, we obtain

\[ (9) \quad e_{n+1} \leq e_n + h(L(e_n + e_n^*) + L,e_n^{**}) + h^2 B/2. \]

Equation (5) implies that \( g_n = t_{q(n)} + hr(n) \), and hence, in a similar manner, we have (after replacing \( n \) by \( (n + 1) \))

\[ (10) \quad e_{n+1}^* \leq ML,e_{n+1} + e_{q(n+1)}^* \]

\[ + hr(n + 1)[L(e_{q(n+1)} + e_{q(n+1)}^* + L,e_{q(n+1)}^{**})] + h^2 r(n + 1)B/2, \]

\[ (11) \quad e_{n+1}^{**} \leq BL,e_{n+1} + L(e_{q(n+1)} + e_{q(n+1)}^* + L,e_{q(n+1)}^{**}) + hr(n + 1)B. \]

We then have two cases to consider:

**Case I.** \( q(n + 1) = n + 1 \) and \( r(n + 1) = 0 \). Under these conditions, (9) is unchanged:

\[ (9a) \quad e_{n+1} \leq e_n(1 + hL) + e_n^* hL + e_n^{**} hL_z + h^2 B/2. \]

(10) becomes

\[ (10a) \quad e_{n+1}^* \leq e_{n+1}(1 + ML_0) = e_{n+1}^* c_0. \]

And (11) becomes

\[ e_{n+1}^{**} \leq (L + BL_0) e_{n+1} + L e_{n+1}^* + L e_{n+1}^{**} \]

or

\[ (11a) \quad e_{n+1}^{**} \leq \left( L + BL_0 + L(1 + ML_0) \right) e_{n+1} = e_{n+1} c_1. \]

Define the partial ordering for vectors: \( v_1 = (v_1^1, \ldots, v_1^k) \leq v_2 = (v_2^1, \ldots, v_2^k) \) if \( v_1^i \leq v_2^i, \ i = 1, \ldots, k \). Then, in vector form, (9a), (10a), and (11a) become

\[
\begin{bmatrix}
  e_{n+1} \\
  e_{n+1}^* \\
  e_{n+1}^{**}
\end{bmatrix}
\leq
\begin{bmatrix}
  1 + hL & hL & hL_z \\
  (1 + hL)c_0 & hLc_0 & hL_zc_0 \\
  (1 + hL)c_1 & hLc_1 & hL_zc_1
\end{bmatrix}
\begin{bmatrix}
  e_n \\
  e_n^* \\
  e_n^{**}
\end{bmatrix}
+ hB
\begin{bmatrix}
  h/2 \\
  hc_0/2 \\
  hc_1/2
\end{bmatrix}
\]

which is of the form \( d_{n+1} \leq A d_n + b_1 \).

**Case 2.** \( q(n + 1) \leq n \) and \( 0 \leq r(n + 1) < 1 \).

Let

\[ \delta_n = \max_{1 \leq i \leq n} e_i, \quad \delta_n^* = \max_{1 \leq i \leq n} e_i^*, \quad \delta_n^{**} = \max_{1 \leq i \leq n} e_i^{**}. \]

Then, (9) becomes

\[ (9b) \quad \delta_{n+1} \leq \delta_n(1 + hL) + \delta_n^* hL + \delta_n^{**} hL_z + h^2 B/2. \]
And (10) becomes
\[ \delta_{n+1} \leq ML_n \delta_{n+1} + \delta_n(1 + hL) + hL \delta_n + hL_c \delta_n + h^2 B/2. \]

Using (9b), we have
\[ \delta_{n+1} \leq (\delta_n(1 + hL) + \delta_n hL + h^2 B/2)(1 + ML_n) \]
or
\[ \delta_{n+1} \leq \delta_n(1 + hL)c_0 + \delta_n hLc_0 + h^2 c_0 B/2. \]

Finally, (11) becomes
\[ \delta_{n+1} \leq \delta_n(1 + hL)c_0 + \delta_n hLc_0 + h^2 c_0 B/2. \]

Further, enlarging \( \delta_n \) to \( \delta_{n+1} \) and \( \delta_n^* \) to \( \delta_{n+1}^* \) on the right, and using \( 1 - L_s > 0 \), we find
\[ \delta_{n+1}^* \leq \delta_n^* \left( \frac{L + BL_c}{1 - L_s} \right) + \delta_n^* \frac{L}{1 - L_s} + \frac{hB}{1 - L_s}. \]

Using (9b) and (10b), we have
\[ \delta_{n+1}^* \leq \delta_n^* \left( \frac{L + BL_c}{1 - L_s} \right) + \delta_n^* \frac{L}{1 - L_s} + \frac{hB}{1 - L_s} \]
or
\[ \delta_{n+1}^* \leq \delta_n(1 + hL)c_1 + \delta_n hLc_1 + h^2 c_1 B/2. \]

Then, as a vector system, (9b), (10b), and (11b) become
\[ \begin{bmatrix} \delta_{n+1} \\ \delta_{n+1}^* \\ \delta_{n+1}^{**} \end{bmatrix} \leq \begin{bmatrix} 1 + hL & hL & hLc_0 \\ (1 + hL)c_0 & hLc_0 & hL, c_0 \\ (1 + hL)c_1 & hLc_1 & hL, c_1 \end{bmatrix} \begin{bmatrix} \delta_n \\ \delta_n^* \\ \delta_n^{**} \end{bmatrix} + \begin{bmatrix} h/2 \\ hB \\ hc_1/2 + 1/(1 - L_s) \end{bmatrix}. \]

which is of the form \( d_{n+1} \leq A_d d_n + b \). Comparing this with the result obtained in Case 1, we find that \( A_1 \) and \( A_2 \) are identical and that \( b_1 \leq b_2 \). Thus, any bound obtained here in Case 2 for \( d_{n+1} \) will also bound \( d_{n+1} \) in Case 1.

To complete the proof, we shall use the following lemmas [3] which may be verified by induction:

**Lemma 1.** Suppose \( A \) is a \( k \times k \) real matrix and \( b \) is a real \( k \)-vector. Let \( \{d_n\} \) \( (n = 0, 1, \cdots) \) satisfy \( d_{n+1} \leq Ad_n + b \). Then
\[ d_{n+1} \leq A^{n+1}d_0 + \left( \sum_{i=0}^{n} A^i \right)b. \]

**Lemma 2.** Let \( p = (p_1, \cdots, p_k) \), \( q = (q_1, \cdots, q_k) \). Suppose the \( k \times k \) matrix \( A \) has the form \( A = p^Tq \). Then
\[ A^n = \left( \sum_{i=1}^{k} p_i q_i \right)^{n-1} A. \]
By Lemma 1,
\[ d_{n+1} \leq A_n^{11} d_0 + \left( \sum_{i=0}^{n} A_i^2 \right) b_2, \]
where
\[ d_0 = \begin{bmatrix}
  c_0 \\
  c_0^* \\
  c_1
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  |z_0 - u_0|
\end{bmatrix}. \]

Then, because
\[ A_2 = \begin{bmatrix}
  1 \\
  c_0 (1 + hL, hL, hL), \\
  c_1
\end{bmatrix}, \]
we can make use of Lemma 2 to obtain
\[ A_2' = (1 + hL + hLc_0 + hLc_1)^{-1} A_2 = (1 + hs)^{-1} A_2. \]

Two results follow from this: \( A_n^{11} = (1 + hs)^n A_2 \leq e^x (h-s^{-1}) A_2 \), and
\[
\sum_{i=1}^{n} A_i = A_2 \sum_{i=1}^{n} (1 + hs)^{i-1} = \frac{(h+hs)^n - 1}{hs} A_2 \leq \frac{1}{hs} (\exp s(b-a) - 1) A_2.
\]

Finally,
\[
d_{n+1} \leq A_n^{11} d_0 + \left( \sum_{i=0}^{n} A_i^2 \right) b_2
\leq h |z_0 - u_0| L_1 e^{1(b-a)} \begin{bmatrix}
  1 \\
  c_0 \\
  c_1
\end{bmatrix}
+ \frac{B}{2s} \left( hs + \frac{1}{1 - L_1} \right) \left( e^{1(b-a)} - 1 \right) \begin{bmatrix}
  1 \\
  c_0 \\
  c_1
\end{bmatrix}
+ B \begin{bmatrix}
  \frac{h}{2} \\
  \frac{hc_0}{2} \\
  \frac{hc_1}{2} + \frac{1}{1 - L_1}
\end{bmatrix},
\]
which gives
\[
c_{n+1} \leq \delta_{n+1} \leq h \left[ |z_0 - u_0| L_1 e^{1(b-a)} + \frac{B}{2s} \left( hs + \frac{1}{1 - L_1} \right) \left( e^{1(b-a)} - 1 \right) + \frac{hB}{2} \right]
\]
and the theorem follows.
For the simplified algorithm, where \( y_n = x_{q(n)} \) the following bound is possible:

\[
d_{n+1} \leq h \left| z_0 - u_0 \right| L_s e^{x(b-a)} + \left( \frac{B}{2s} \left( h_s + \frac{1}{1 - L_s} \right) + \frac{1}{s} \left( \frac{ML}{1 - L_s} \right) \right) e^{x(b-a)} - 1 \left[ \begin{array}{c} 1 \\ c_0 \\ c_1 \end{array} \right]
\]

(13)

\[
+ \left( \frac{B}{2s} \left( h_s + \frac{1}{1 - L_s} \right) + \frac{1}{s} \left( \frac{ML}{1 - L_s} \right) \right) e^{x(b-a)} - 1 \left[ \begin{array}{c} 1 \\ c_0 \\ c_1 \end{array} \right]
\]

\[
+ B \left[ \begin{array}{c} h \\ \frac{hc_0}{2} \\ \frac{hc_1}{2} + \frac{1}{1 - L_s} \end{array} \right] + \left[ \begin{array}{c} 0 \\ M \\ \frac{ML}{1 - L_s} \end{array} \right],
\]

and hence

\[
e_{n+1} \leq h \left| z_0 - u_0 \right| L_s e^{x(b-a)} + \left( \frac{B}{2s} \left( h_s + \frac{1}{1 - L_s} \right) + \frac{1}{s} \left( \frac{ML}{1 - L_s} \right) \right) e^{x(b-a)} - 1 + \frac{hB}{2} \right).
\]

**Table I.** \( x_n(h) \) denotes the value of \( x_n \) for step size \( h \).

<table>
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<th>( x(t_n) )</th>
<th>( x_n(2^{-4}) )</th>
<th>( x_n(2^{-6}) )</th>
<th>( x_n(2^{-8}) )</th>
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Table II. \( x^{(1)}(h) \) denotes the value of \( x_n \) for step size \( h \) by algorithm \&; \( x^{(2)}(h) \) denotes the value of \( x_n \) for step size \( h \) by the simplified algorithm.

<table>
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4. Examples. (a) We solve the IVP

\[
x'(t) = \frac{-4tx^2(t)}{4 + \log^2 \cos t} + \tan 2t + \frac{1}{2} \tan^{-1} z
\]

\((z_0 = 0, x_0 = 0, z = x'(g(t, x(t))) = x'(tx^2(t))/(1 + x^2(t)))\) on the interval \([0, .75]\).

The existence and uniqueness of the solution is guaranteed by the results of [2] mentioned earlier. The only solution is \( x(t) = -\frac{1}{2} \log \cos 2t \).

The results of the computation by algorithm \& are given in Table I.

(b) Consider the IVP

\[
x'(t) = \cos t(1 + y) + xz - \sin(t(1 + \sin^2 t)),
\]

with \( y = x'(tx^2(t)), z = x'(tx^2(t)), z_0 = 1, x_0 = 0, \) on the interval \([0, 1]\). As in example (a), existence and uniqueness of the solution are guaranteed by the results of [2]. Here, the solution is \( x(t) = \sin t \).

The results of the computation by the algorithm \& and by the simplified algorithm are given in Table II.