MATHEMATICS OF COMPUTATION, VOLUME 27, NUMBER 124, OCTOBER 1973

The First Occurrence of Large Gaps Between Successive Primes

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Abstract. A table of the first occurrence of a string of $2r - 1$ composite numbers between two primes is given for $r = 158(1)267, 269, 270, 273, 275, 276, 281, 282, 291, 294$ and 301. All such strings between primes less than 2.6×10^{12} have been accounted for. The computation supports some conjectures on the distribution of these strings.

1. Introduction. Let $p_1 = 2, p_2 = 3, \cdots$ be the sequence of primes. For positive integers r, define

$$f(r) = p_j \quad \text{if } j \text{ is the least positive index such that } p_{j+1} - p_j = 2r,$$
$$= \infty \quad \text{if no such } j \text{ exists,}$$

and

$$g(r) = p_k \quad \text{if } k \text{ is the least positive index such that } p_{k+1} - p_k \geqq 2r.$$

Very little is known about the functions f and g. It has not even been established that $f(r)$ is finite for all $r \geqq 1$. Certainly,

$$F(r) \geqq G(r) \geqq c_1 \log r \quad \text{and} \quad G(r) \leqq c_2 r$$

for positive constants c_1 and c_2, where $F(r) = \log f(r)$ and $G(r) = \log g(r)$. For these and other known results, see Johnson [8], Prachar [12], and Shanks [13].

Although the rigorous results are weak, heuristic probability arguments and empirical evidence suggest some plausible conjectures; see [1], [3]–[7], [10], [11], [13], and [15]. We give some empirical evidence which suggests that the function

$$\phi(r) = \frac{F(r) - (2r)^{1/2}}{\log(2r)}$$

is bounded, which implies that $|F(r) - (2r)^{1/2}|$ and $|G(r) - (2r)^{1/2}|$ are $O(\log r)$ as $r \to \infty$.

Lander and Parkin [10] have computed $f(r)$ for all values of r such that $f(r) \leqq 1.096 \times 10^{10}$. In Section 2, we describe an extension of their computations. Since $g(r) = \min_{s \geqq r} f(s)$, it is sufficient to tabulate $f(r)$. Table 1 gives $f(r)$ for all $r \geqq 158$ such that $f(r) \leqq 2.6 \times 10^{12}$ (for $r = 1, \cdots, 157$ see [10], where the notation $D = 2r$ and $P_a = f(r)$ is used). In Table 1 the "maximal" gaps (i.e., those for which $f(r) = g(r)$) are marked with an asterisk.

Received September 18, 1972.

AMS (MOS) subject classifications (1970). Primary 10-04, 10A20, 10A25, 10A40, 65A05.

Key words and phrases. Prime, distribution of primes, prime gap, maximal prime gap, successive composites, consecutive primes.

TABLE 1

The First Occurrence of 2r — 1 Consecutive Composite Numbers

r	$f(r)$	r	$f(r)$
158	12109172293	198	50806025873
159	4372999721	199	40267027589
160*	2300942549	200	47203303159
161	7961074441	201	44293346177
162	10958687879	202	144999022043
163	5837935373	203	49306638307
164	13086861181	204	134664608389
165	6291356009	205	98276144093
166	5893180121	206	124221464119
167	30827138509	207	49914935177
168*	3842610773	208	121972158437
169	22076314313	209	129300694603
170	8605261447	210	82490815123
171	12010745569	211	280974865361
172	19724087267	212	264495345259
173	11291401837	213	180265084403
174	17002876643	214	219950168411
175	16808773277	215	250964194171
176	30750892801	216	87241770619
177*	4302407359	217	127084569923
178	24355072517	218	367459059871
179	16792321339	219	101328529441
180	20068818197	220	141846299801
181	35877724601	221	417470554687
182	25425617317	222	36172730063
183	20108776097	223	190418076203
184	51430518413	224	402872474743
185	59942358571	225	63816175447
186	20404137779	226	466855187471
187	23064761663	227	202530831163
188	16161669787	228*	25056082087
189	38116957819	229	304040251469
190	23323808741	230	131956235563
191*	10726904659	231	400729567081
192*	20678048297	232*	42652618343
193	35238645587	233	565855695631
194	156798792223	234*	127976334671
195	53241805651	235	681753256133
196	117215204531	236	865244709607
197*	22367084959	237*	182226896239

TABLE 1 (*continued*)

r	$f(r)$	r	$f(r)$
238	725978934347	258*	416608695821
239	367766547571	259	2296497058133
240	482423533897	260	2336167262449
241	1051602787181	261	1214820695701
242	767644374817	262	2256065636039
243*	241160624143	263	1620505682371
244	1275363152099	264	1529741785139
245*	297501075799	265	2205492372371
246	910361180689	266*	461690510011
247	804541404419	267*	614487453523
248	880318998907	269	2122536905311
249	428315806823	270*	738832927927
250*	303371455241	273	2164206784721
251	1258535916601	275	2496646209271
252	747431049203	276	2210401546601
253	1339347750707	281	2081209441279
254	1841086484491	282	1480064231153
255	2209016910131	291*	1346294310749
256	1999066711391	294*	1408695493609
257*	304599508537	301*	1968188556461

The heuristic argument of Cadwell [3] suggests that $\phi(r) \simeq \frac{1}{4}$, but our computations indicate that $\phi(r)$ is usually in $(\frac{1}{2}, 1)$. Table 2 gives those values of r for which $f(r) \leqq 2.6 \times 10^{12}$ and either $\phi(r) \leqq \frac{1}{2}$ or $\phi(r) \geqq 1$.

In number-theoretic computations it is often necessary to store blocks of consecutive primes. To save storage space, it may be convenient to store the semidifferences $\delta_i = (p_{i+1} - p_i)/2$. Our computations show that $\delta_i < 256$, so only eight bits of storage are required for each δ_i, provided $p_i \leqq 3 \times 10^{11}$.

The largest gap found* was one of 601 composite numbers following the prime 1968188556461. From Brent [2], Jones, Lal and Blundon [9], and Weintraub [14], it is clear that strings of more than 400 consecutive composites between primes less than 10^{17} are extremely rare. For example, between 10^{16} and $10^{16} + 10^8$ there are 2714904 primes, but only 15 gaps of more than 400 consecutive composites (the largest gap being of 531 composites).

2. Computation of Prime Gaps. To compute Table 1, integers up to 2.6×10^{12} were sieved for primes in blocks of length $16336320 = 2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$. A one-bit flag represented each odd number in the block, so 1021020 bytes (each eight bits) of storage were required for the sieve. After setting all bits to zero, the bits corresponding to numbers divisible by 3, 5, 7, 11, $\cdots$ were set to one. (This required the primes up to

* Added in proof. In extending the search to 2.686×10^{12} we found $f(272) = 2652427555639$, $f(326) = 2614941710599$, and $\phi(326) = 0.4719$.

TABLE 2

Values of $\phi(r)$ Lying Outside the Interval $(\frac{1}{2}, 1)$

r	$\phi(r)$	r	$\phi(r)$
1	−0.4553	58	1.0120
2	−0.0390	74	0.4700
3	0.3829	79	1.0159
7	0.3735	83	1.0484
8	1.2669	100	1.0585
17	0.3856	105	0.4415
19	1.1447	114	1.0155
23	1.1820	167	1.0112
36	0.4371	194	1.0200
37	1.0010	228	0.4231
56	0.4745	232	0.4781

$(2.6 \times 10^{12})^{1/2}$, which had been precomputed and stored as described in Section 1.) The remaining zero bits corresponded to primes, and these were efficiently scanned for large gaps by an algorithm like the first algorithm of [10].

The computations were performed on an IBM 360/91 computer at the IBM T. J. Watson Research Center, and various machine-dependent coding tricks were used to speed up the sieving and scanning. For example, with our choice of sieve size it was possible to set eight bits, corresponding to eight numbers divisible by 3, 5, 7, 11, 13 or 17, with just one MVI (move immediate) instruction. The program ran in 1194K bytes and required less than 7.2 seconds for each block of numbers near 2.6×10^{12} (i.e., less than 4.4×10^{-7} seconds per number).

The computed values of $f(r)$ were checked by a program which verified in a straightforward way that $f(r)$ and $f(r) + 2r$ were prime and all the intermediate integers were composite. Our results confirm those of Lander and Parkin [10], and it is unlikely that a machine or programming error could have resulted in a large gap being overlooked.

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