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On the Conditional Equivalence of Two Starting Methods for the Second Algorithm of Remez

By R. E. Huddleston

Abstract. In computing best min-max rational approximations by the second algorithm of Remez (which is an iterative procedure), one must provide a starting approximation. A method proposed by Ralston and one by Werner are shown to be equivalent under reasonable conditions.

1. Introduction. Ralston [3, p. 286] and Werner, Stoer, and Bommas [4] (hereinafter referred to as Werner) have suggested methods for obtaining a starting approximation for the Remez algorithm for rational Chebyshev approximation which work well in all but the hardest cases. (Both Ralston and Werner have also proposed methods for the remaining cases but those results will not be discussed here.) Werner's method is much easier to implement than Ralston's and, as we shall show here, produces the same starting approximation under very reasonable conditions.

2. Werner's Method. Let $f \in C[a, b]$ be the function which is to be approximated by a rational function of degree m in the numerator and n in the denominator. Denote the starting approximation by

$$R_{m,n}^{(0)}(x) = \frac{P_m^{(0)}(x)}{Q_n^{(0)}(x)} = \frac{\sum_{j=0}^{m} a_j T_j(x)}{\sum_{j=0}^{n} b_j T_j(x)}, \tag{2.1}$$

where $T_r(x) = \cos(r \arccos x)$ is the Chebyshev polynomial of degree r.

Without loss of generality we assume that $[a, b] = [-1, 1]$ and that $b_0 = 1$. To determine the $m + n + 1$ coefficients a_i, b_i Werner suggests the rational interpolation

$$R_{m,n}^{(0)}(Z_i) = f(Z_i), \qquad i = 0, 1, \cdots, m + n + 1, \tag{2.2}$$

where the Z_i are the zeroes of the Chebyshev polynomial of degree $m + n + 1$ which are given by

$$Z_i = \cos((2i + 1)\pi/2(m + n + 1)). \tag{2.3}$$

3. Ralston's Method. Obtaining the starting approximation (2.1) by Ralston's

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method is a two part procedure. First one must determine the coefficients C_j in the Chebyshev expansion

$$f(x) \cong \tfrac{1}{2}C_0 + \sum_{j=1}^{m+2n} C_j T_j(x) \tag{3.1}$$

where

$$C_j = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_j(x)}{(1-x^2)^{1/2}}\, dx, \qquad j = 0, 1, \cdots, m + 2n. \tag{3.2}$$

Then one must solve the linear system

$$\begin{aligned} a_0 &= \tfrac{1}{2} \sum_{j=0}^{n} b_j C_j, \\ a_r &= \tfrac{1}{2} \sum_{j=0}^{n} b_j (C_{|r-j|} + C_{r+j}), \qquad r = 1, \cdots, m + n, \end{aligned} \tag{3.3}$$

where $a_r = 0$ for $r > m$.

If we express f as

$$f(x) = \tfrac{1}{2}C_0 + \sum_{j=1}^{\infty} C_j T_j(x)$$

then we may write

$$f(x) - R_{m,n}^{(0)}(x) = N(x) \Big/ \sum_{j=0}^{n} b_j T_j(x)$$

where

$$N(x) = \left[\frac{1}{2} C_0 + \sum_{j=1}^{\infty} C_j T_j(x)\right]\left[\sum_{j=0}^{n} b_j T_j(x)\right] - \sum_{j=0}^{m} a_j T_j(x).$$

Requiring that the coefficients of $T_j(x)$ in $N(x)$ vanish for $j = 0, 1, \cdots, m + n$ leads to the linear system (3.3).

4. Conditional Equivalence. In Ralston's procedure one must evaluate the integrals (3.2). The solution of (3.3) requires $m + 2n$ of the Chebyshev coefficients C_j. This is as opposed to the evaluation of f at only $m + n + 1$ points in Werner's method. Thus, on the surface these two linear systems, (2.2) and (3.3), do not seem equivalent. In order to establish a connection between them, we shall use two known identities concerning Chebyshev polynomials (see [2, p. 215]):

$$\begin{aligned} \sum_{i=0}^{m+n} T_r(Z_i)T_j(Z_i) &= 0, & 0 \leqq r \neq j \leqq m + n, \\ &= (m + n + 1)/2, & 0 < r = j \leqq m + n, \\ &= m + n + 1, & 0 = r = j, \end{aligned} \tag{4.1}$$

where the Z_i are given by (2.3), and

$$T_{r+j}(x) + T_{|r-j|}(x) = 2T_r(x)T_j(x). \tag{4.2}$$

Let us rewrite the linear system (2.2) for Werner's approximation as

$$\sum_{j=0}^{m} a_j T_j(Z_i) = \sum_{j=0}^{n} b_j f(Z_i) T_j(Z_i).$$

Multiplying both sides by $T_r(Z_i)$ and summing over i we have

$$\sum_{j=0}^{m} \left(a_j \sum_{i=0}^{m+n} T_j(Z_i) T_r(Z_i) \right) = \sum_{j=0}^{n} \left(b_j \sum_{i=0}^{m+n} f(Z_i) T_j(Z_i) T_r(Z_i) \right).$$

Using (4.1) we have

$$\sum_{j=0}^{m} \left(a_j \delta_{jr} \begin{bmatrix} (m+n+1)/2, & j = r > 0 \\ m+n+1, & j = r = 0 \end{bmatrix} \right) = \sum_{j=0}^{n} \left(b_j \sum_{i=0}^{m+n} f(Z_i) T_j(Z_i) T_r(Z_i) \right)$$

where δ_{jr} is the Kronecker delta function. For $r = 0$ this yields

$$a_0 = \sum_{j=0}^{n} b_j \left(\frac{1}{m+n+1} \sum_{i=0}^{m+n} f(Z_i) T_j(Z_i) T_0(Z_i) \right).$$

For $1 \leqq r \leqq m$, we have

$$a_r = \sum_{j=0}^{n} b_j \left(\frac{2}{m+n+1} \sum_{i=0}^{m+n} f(Z_i) T_j(Z_i) T_r(Z_i) \right),$$

and, for $r > m$,

$$0 = \sum_{j=0}^{n} b_j \left(\frac{2}{m+n+1} \sum_{i=0}^{m+n} f(Z_i) T_j(Z_i) T_r(Z_i) \right).$$

If we now employ (4.2) we have

$$a_0 = \frac{1}{2} \sum_{j=0}^{n} b_j \left(\frac{2}{m+n+1} \sum_{i=0}^{m+n} f(Z_i) T_j(Z_i) \right)$$

and

$$\begin{aligned} a_r = \frac{1}{2} \sum_{j=0}^{n} b_j \Bigg(\left(\frac{2}{m+n+1} \sum_{i=0}^{m+n} f(Z_i) T_{r+j}(Z_i) \right) \\ + \left(\frac{2}{m+n+1} \sum_{i=0}^{m+n} f(Z_i) T_{|r-j|}(Z_i) \right) \Bigg) \end{aligned} \tag{4.3}$$

for $1 \leqq r \leqq m + n$ where $a_r = 0$ for $r > m$. We note that (4.3) looks a bit more like (3.3) now. In fact, they would be equal if the C_k, as defined by (3.2), were equal to $(2/(m + n + 1)) \sum_{i=0}^{m+n} f(Z_i)T_k(Z_i)$. Using the Gauss-Chebyshev quadrature rule (see [3, p. 99]), we have

$$C_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{(1-x^2)^{1/2}} dx = \frac{2}{m+n+1} \sum_{i=0}^{m+n} f(Z_i) T_k(Z_i) + E,$$

where

$$E = \frac{1}{2^{2(m+n)}(2m+2n)!} \frac{d^{(2m+2n+2)}}{dx^{(2m+2n+2)}} (f(\eta) T_k(\eta)). \tag{4.4}$$

Thus, under the condition that the Gauss-Chebyshev quadrature rule with $m + n + 1$ points is used, the two methods of obtaining a starting approximation are equivalent. In practice one would find it rare to be able to exactly evaluate integrals such as (3.2)—thus some kind of quadrature rule must be used. Keeping in mind that we are trying to derive a starting approximation for an iterative procedure, the error given by E seems sufficiently small.

A FORTRAN implementation of the ALGOL procedure by Werner, Stoer, and Bommas [4] and a document [1] describing the implementation is available from the author.

Numerical Applications Division 8441
Sandia Laboratories
Livermore, California 94550

1. R. E. Huddleston, *REHRAT—A Program for Best Min-Max Rational Approximation*, Report #(SCL-DR-72-370), Sandia Laboratories, Livermore, California, 1972.
2. E. Isaacson & H. B. Keller, *Analysis of Numerical Methods*, Wiley, New York, 1966. MR **34** #924.
3. A. Ralston, *A First Course in Numerical Analysis*, McGraw-Hill, New York, 1965. MR **32** #8479.
4. H. K. E. Werner, J. Stoer & W. Bommas, "Rational Chebyshev approximation," *Numer. Math.*, v. 10, 1967, pp. 289–306.