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Error Analysis of a Computation of Euler's Constant*

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Abstract. A complete error analysis of a computation of γ, Euler's constant, is given. The results have been used to compute γ to 7114 places and this value has been deposited in the UMT file.

1. Introduction. In a paper on ergodic computations with continued fractions [1], we used 3561 decimal places of γ, Euler's constant, as given by Sweeney [7] to compute 3420 partial quotients of the continued fraction expansion of γ. The partial quotients were sent to the Unpublished Manuscript Tables file and were there compared by Dr. Wrench with those given by Choong et al. [3]. Some disagreements were found and it was eventually decided to recompute Sweeney's value. This involved a careful reading of Sweeney's method and, as his error analysis is not detailed, a distinct error analysis resulted. This analysis is presented here.

2. Error Analysis. We begin with the exponential integral $-\mathrm{Ei}(-x)$ [2, p. 334], and we consider only $x > 1$:

$$-\mathrm{Ei}(-x) = \int_x^\infty \frac{e^{-t}}{t}\,dt = -\gamma - \ln x + S(x), \tag{1}$$

where

$$S(x) = x - \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} - \frac{x^4}{4\cdot 4!} + \cdots .$$

The analysis of [6, p. 26] can be adapted to show

$$\int_x^\infty \frac{e^{-t}}{t}\,dt = \frac{e^{-x}}{x}\left(1 - \frac{1!}{x} + \frac{2!}{x^2} - \cdots + \frac{(-1)^n n!}{x^n} + R_n(x)\right),$$

where $|R_n(x)| \leqq (n+1)!/x^{n+1}$. However, we only require $n = 0$ and it is easy to see that

$$xe^x \int_x^\infty \frac{e^{-t}}{t}\,dt = \int_0^\infty \frac{e^{-s}}{1 + s/x}\,ds = 1 - \int_0^\infty \frac{s/x}{1 + s/x} e^{-s}\,ds = 1 + R_0(x).$$

Since, for $x > 0$,

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$$0 < \int_0^\infty \frac{s/x}{1 + s/x} e^{-s}\, ds < \frac{1}{x} \int_0^\infty s e^{-s}\, ds = \frac{1}{x},$$

we infer that

$$\frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} \leqq \int_x^\infty \frac{e^{-t}}{t}\, dt \leqq \frac{e^{-x}}{x}. \tag{2}$$

By Eqs. (1) and (2),

$$S(x) - \frac{e^{-x}}{x} - \ln x \leqq \gamma \leqq S(x) + \frac{e^{-x}}{x^2} - \frac{e^{-x}}{x} - \ln x. \tag{3}$$

Our problem is to use Eq. (3) to compute γ to a desired number of decimal places. After x is taken to be a power of 2, we must approximate e^{-x}/x, ln 2, and $S(x)$. The computation was done on the Maniac II computer which does multiple-precision integer arithmetic without special programming. Therefore each function above will be multiplied by an appropriate power of 10, say 10^α. Of the α places in our answer, we will require that each answer be correct to $d - 1$ places. Equation (3) becomes

$$\begin{aligned} 10^\alpha S(x) - 10^\alpha \frac{e^{-x}}{x} - 10^\alpha \ln x \leqq 10^\alpha\gamma \leqq 10^\alpha S(x) \\ + 10^\alpha \frac{e^{-x}}{x^2} - 10^\alpha \frac{e^{-x}}{x} - 10^\alpha \ln x. \end{aligned} \tag{4}$$

We first consider the error in the exponential terms of (4).

$$\left| 10^\alpha \left(\frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} \right) \right| \leqq 10^\alpha \frac{e^{-x}}{x} < 10^{\alpha - x/\ln 10}.$$

If the exponential terms are neglected in (3) and we desire $d - 1$ correct places, we must have $\alpha - x/\ln 10 < \alpha - d$ or $d \ln 10 < x$. Thus, we determine d from

$$d = [x/\ln 10]. \tag{5}$$

The following procedure is used to approximate $S(x)$. Let

$$A_{n-1} = 10^\alpha - \frac{n-1}{n^2} x,$$

$$A_k = 10^\alpha - \frac{k}{(k+1)^2} x A_{k+1}, \qquad 1 \leqq k < n - 1.$$

Then define $T(x)$ by

$$\begin{aligned} 10^\alpha T(x) &= x A_1 = x\left(10^\alpha - \frac{x}{2^2} A_2\right) \\ &= x\left(10^\alpha - \frac{x}{2^2}\left(10^\alpha - \frac{2x}{3^2} A_3\right)\right) \\ &\cdots \\ &= 10^\alpha\left(x - \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} - \frac{x^4}{4\cdot 4!} + \cdots + (-1)^{n+1} \frac{x^n}{n\cdot n!}\right). \end{aligned}$$

The truncation error in using $10^\alpha T(x)$ in place of $10^\alpha S(x)$ is

$$|10^\alpha S(x) - 10^\alpha T(x)|$$
$$\leqq 10^\alpha\left(\frac{x^{n+1}}{(n+1)(n+1)!} + \frac{x^{n+2}}{(n+2)(n+2)!} + \frac{x^{n+3}}{(n+3)(n+3)!} + \cdots\right)$$
$$\leqq \frac{10^\alpha}{n+1}\left(\frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \frac{x^{n+3}}{(n+3)!} + \cdots\right).$$

The quantity in parentheses is the remainder term in the Taylor expansion of e^x and is therefore equal to $x^{n+1}e^{\theta x}/(n+1)!$, $\theta \in (0, 1)$. Next, we assume $n > 2x$ and use a technique of Courant [4, p. 326] to obtain $x^{n+1}/(n+1)! < (2x)^{2x}/(2x)!2^{-n-1}$. Thus

$$|10^\alpha S(x) - 10^\alpha T(x)| < \frac{10^\alpha}{n+1} e^x \left(\frac{(2x)^{2x}}{(2x)!} 2^{-n-1}\right).$$

Using the fact that Stirling's formula underestimates $(2x)!$ [5, p. 54], we obtain

$$|10^\alpha S(x) - 10^\alpha T(x)| < \frac{10^\alpha}{n+1} \frac{e^{3x-(n+1)\ln 2}}{(2\pi)^{1/2}(2x)^{1/2}} < 10^{\alpha+(3x-(n+1)\ln 2)/\ln 10}.$$

Since we require $d - 1$ correct places, we take

$$\alpha + (3x - (n+1)\ln 2)/\ln 10 < \alpha - d$$

which yields

$$n > d \ln 10/\ln 2 + 3x/\ln 2 - 1.$$

But we have $x > d \ln 10$, so it is sufficient to take

$$(6) \qquad n = [4x/\ln 2].$$

We note that $n = [4x/\ln 2] > 2x$ as required above.

There is also round-off error in computing $10^\alpha T(x)$. Assume that an error of ϵ_k is made in the kth iteration:

$$A_k = 10^\alpha - \frac{kx}{(k+1)^2} A_{k+1} + \epsilon_k.$$

Then

$$10^\alpha \hat{T}(x) = xA_1$$
$$\cdots$$
$$= 10^\alpha T(x) + \left(x\epsilon_1 - \frac{x^2}{2\cdot 2!}\epsilon_2 + \frac{x^3}{3\cdot 3!}\epsilon_3 - \cdots + \frac{(-1)^{n+1}x^n}{n\cdot n!}\epsilon_n\right).$$

If we assume $|\epsilon_k| \leqq \epsilon = 1$ for all k, then

$$|10^\alpha \hat{T}(x) - 10^\alpha T(x)| < e^x.$$

Now $e^x = 10^{x/\ln 10} < 10^{\alpha-d}$ if

$$(7) \qquad \alpha = 2d + 1.$$

If one has ln 2 for sufficiently many decimal places, one can use (3) to compute γ to the desired number of decimals. The computation of the decimals of ln 2 is discussed in the next section.

3. Computation of ln 2. Choose β to be some positive integer to be determined. The following series is used (it can be obtained from Taylor's series):

$$10^\beta \ln 2 = 2\left(\frac{10^\beta}{3} + \frac{10^\beta}{3\cdot 3^3} + \frac{10^\beta}{5\cdot 3^5} + \frac{10^\beta}{7\cdot 3^7} + \cdots\right).$$

This series was approximated by

$$A = 2\left(\left[\frac{10^\beta}{3}\right] + \left[\frac{10^\beta}{3\cdot 3^3}\right] + \left[\frac{10^\beta}{5\cdot 3^5}\right] + \cdots + \left[\frac{10^\beta}{(2(k-1)+1)3^{2(k-1)+1}}\right]\right)$$

where k is determined automatically for fixed β by the condition that

$$10^\beta < (2k+1)3^{2k+1}. \tag{8}$$

Then

$$\begin{aligned} 0 &\leqq 10^\beta \ln 2 - A \\ &= 2\left\{\left(\frac{10^\beta}{3} - \left[\frac{10^\beta}{3}\right]\right) + \left(\frac{10^\beta}{3\cdot 3^3} - \left[\frac{10^\beta}{3\cdot 3^3}\right]\right)\right. \\ &\qquad \left. + \cdots + \left(\frac{10^\beta}{(2(k-1)+1)3^{2(k-1)+1}} - \left[\frac{10^\beta}{(2(k-1)+1)3^{2(k-1)+1}}\right]\right)\right\} \\ &\quad + 2\cdot 10^\beta\left(\frac{1}{(2k+1)3^{2k+1}} + \frac{1}{(2k+3)3^{2k+3}} + \cdots\right). \end{aligned} \tag{9}$$

The term outside the curly brackets in (9) is dominated by

$$\frac{2\cdot 10^\beta}{(2k+1)3^{2k+1}} \sum_{l=0}^{\infty} \frac{1}{9^l} = \frac{2\cdot 10^\beta}{(2k+1)} \frac{1}{3^{2k+1}} \frac{9}{8} \leqq \frac{9}{4},$$

making use of (8). The curly brackets in (9) are dominated by $k - 1$. Hence

$$0 \leqq 10^\beta \ln 2 - A \leqq 2(k-1) + 9/4, \tag{10}$$

where k is determined as the least k which satisfies (8). An upper bound to k as given by (8) is

$$\frac{\beta}{2} \frac{\ln 10}{\ln 3}. \tag{11}$$

Hence

$$0 \leqq 10^\beta \ln 2 - A \leqq \beta \ln 10/\ln 3 + \tfrac{1}{4}. \tag{12}$$

In our computation, we choose $\beta = 7140$. One sees from (12) that the error in ln 2 as given by A is in the last 5 places in the 7140 places. We actually have only reported and used 7121 places.

TABLE 1

n	Sample Frequency of n	Theoretical Frequency of n: $\frac{1}{\ln 2}\ln\frac{(n+1)^2}{n(n+2)}$
1	0.4225	0.4150
2	0.1646	0.1699
3	0.0896	0.0931
4	0.0527	0.0589
5	0.0438	0.0406
6	0.0308	0.0297
7	0.0228	0.0227
8	0.0216	0.0179
9	0.0121	0.0144
10	0.0124	0.0119

TABLE 2

x	Guaranteed Number of Correct Digits $d - 1$	Actual Number of Correct Digits
8	2	4
16	5	7
32	12	14
64	26	29
128	54	57
256	110	113
512	221	224
1024	443	446
2048	888	889
4096	1777	1795
8192	3556	3561
16384	7114	—

4. Computation of γ. For our calculation, we used $x = 2^{14}$. From this x, we obtained $d = [x/\ln 10] = 7115$, $\alpha = 2d + 1 = 14231$, $n = [4x/\ln 2] = 94548$, and $k = [\alpha \ln 10/(2 \ln 3)] + 1 = 14914$. The above analysis shows that our computation of γ is accurate to 7114 places. The errors from e^{-x}/x and $S(x)$ might each affect the 7115th place.

From this computation, we obtained 7114 correct decimal places of γ. These values were used to calculate 6920 partial quotients in the continued fraction expansion of γ. The 7121 places of ln 2 yielded 6890 partial quotients of ln 2. Note that the number of partial quotients of γ is more than that of ln 2. These have been sent to the Unpublished Manuscript Tables (UMT) file of this journal.

In Choong et al. [3], Table 1 gives sample frequency of n and theoretical frequency of n for 3470 partial quotients of γ. Our Table 1 corrects their Table 1. Our Table 2 gives our results for $x = 2^t$ ($t = 3, 4, \cdots, 14$) and is thus a check of our analysis.

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