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Rational Chebyshev Approximations for the Modified Bessel Functions $I_0(x)$ and $I_1(x)$

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Abstract. This note presents nearly-best rational approximations for the functions $I_0(x)$ and $I_1(x)$, with relative errors ranging down to 10^{-23}.

The most useful set of approximations for $I_n(x)$ for $n = 0, 1$ are the Chebyshev series expansions given in [1] and [2]. The expansions in [1] apply to the functions $I_n(x)/x^n$ for the range $|x| \leqq 8$ and to the functions $(2\pi x)^{1/2}e^{-x}I_n(x)$ for the range $x \geqq 8$. The advantage of these expansions is that they can be truncated to give near-minimax approximations of arbitrary accuracy. However, they suffer from two minor defects, namely a loss of two digits of precision by cancellation for small values of x, and a lack of balance in the amount of computation required in the two ranges. For example, to compute 14S approximations, we need 15 terms of the series for $|x| \leqq 8$, and 17 terms together with the computation of $(2\pi x)^{-1/2}$ and e^x for $x \geqq 8$. Shortening the lower range reduces the amount of cancellation [3], but increases the imbalance. If we use rational function approximations and minimize the relative error, we find that we can reduce the cancellation and still increase the lower range, thereby obtaining a better balance.

Other rational function approximations for $I_0(x)$ and $I_1(x)$ are given in [4]. They are limited to nine or ten digits of accuracy and, since they minimize the absolute error, are less efficient than those presented here. A number of rational approximations are also given in [5], but they only apply to the range $|x| \leqq 1$.

This note gives nearly-best rational function approximations for the complete range of the argument, with relative errors ranging down to 10^{-23}. The approximation forms and intervals are

$$I_n(x) \simeq x^n R_{lm}(x^2), \qquad |x| \leqq 15.0,$$

$$I_n(x) \simeq x^{-1/2}e^x R_{lm}(1/x), \qquad x \geqq 15.0,$$

for $n = 0, 1$, where $R_{lm}(x)$ are rational functions of degree l in the numerator and m in the denominator. The details of the approximations are given in the tables that appear in the microfiche section of this issue. The format is similar to that used in [6].

Tables I to IV summarize the best approximations in the L_∞ Walsh arrays of the functions, and Tables V to XXV give the coefficients of selected approximations. The precision is defined as

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$$-\log_{10} \max_x \left| \frac{f(x) - R_{lm}(x)}{f(x)} \right|$$

where $f(x)$ is the function being approximated and the maximum is taken over the appropriate interval. The "cancellation" is a measure of the maximum number of decimal digits lost by cancellation over the range of the approximation. For a polynomial we divide the sum of the terms by the sum of the moduli of the terms. The negative logarithm of the modulus of this ratio gives the cancellation for one argument x. The maximum cancellation over the range is taken as the cancellation of the polynomial. The cancellation of a rational function is the maximum of the cancellations of the numerator and denominator. With this definition a value of 0.48 for cancellation corresponds to a loss of one binary digit, and a value of 0.85 corresponds to two binary digits.

In the range $|x| \leqq 15.0$, the rational functions are expressed in terms of power polynomials of the form

$$P_l(x) = \sum_{j=0}^{l} p_j x^j.$$

In the range $x \geqq 15.0$, the Chebyshev series form is better conditioned, and we use

$$P_l(1/x) = \tfrac{1}{2}p_0 + p_1 T_1(\xi) + p_2 T_2(\xi) + \cdots + p_l T_l(\xi) = \sum_{j=0}^{l}{}' p_j T_j(\xi)$$

where $\xi = 30/x - 1$.

In Tables I to IV, we list where possible the most accurate approximations of degree $l + m$ having cancellations not greater than 0.48. For the range $|x| \leqq 15.0$, the lowest degree approximations all have cancellations greater than 0.48, and we have selected those with the smallest cancellations. The approximations in Tables I and II have precisions only slightly smaller than the maxima of the same degree. The greatest difference in precision between the most accurate approximations and those given in Tables III and IV is about 2.0.

All computations were done on a CDC 6600 in 29 decimal arithmetic, using a version of the second algorithm of Remes due to Ralston [7]. The master routines, based on the standard power series and asymptotic series expansions, were verified to be accurate to at least 27S by comparison with the values in [8] and by means of the Wronskian $I_0(x)K_1(x) + I_1(x)K_0(x) = 1/x$. The approximations in Tables V to XXV were verified by comparing them with the master routines for 5000 pseudorandom values of the argument.

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