

## Evaluation of a Constant Associated with a Parking Problem

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**Abstract.** A constant associated with a random space filling problem is computed to 19D. This is achieved by numerically solving an integral-difference equation.

**1. Introduction.** During the past few years, several mathematicians, A. Rényi [1], A. Dvoretzky and H. Robbins [2] and J. J. A. Beenakker [3], have studied the random process in which cars of length 1 are parked in a street  $[0, x]$  of length  $x \geq 1$ . The centre of cars is a random variable which is uniformly distributed on  $[\frac{1}{2}, x - \frac{1}{2}]$ . Let  $N_x$  denote the number of cars parked so that the distance between any two consecutive cars is less than or equal to 1. The expectation  $M(x) = E(N_x)$  is a continuous function for  $x \geq 1$  and satisfies the integral equation

$$(1) \quad M(x + 1) = \frac{2}{x} \int_0^x M(t) dt + 1, \quad x > 0,$$

together with the initial conditions

$$M(x) = 0, \quad 0 \leq x < 1, \quad M(1) = 1.$$

In the investigations [1], [2], [3], Eq. (1) has been solved to a varying degree of accuracy. These solutions are

$$(2) \quad M(x) = \lambda(x + 1) - 1 + O(x^{-n}), \quad n \geq 1 \quad (\text{Rényi, 1958}),$$

$$(3) \quad M(x) = \lambda(x + 1) - 1 + O\left(\left(\frac{2e}{x}\right)^{x-3/2}\right), \quad x \rightarrow \infty$$

(Dvoretzky and Robbins, 1964),

$$(4) \quad M(x) = \lambda(x + 1) - 1 + O\left(\left(\frac{2e}{x \log x}\right)^x\right), \quad x \rightarrow \infty \quad (\text{Beenakker, 1966}),$$

where

$$\lambda = \int_0^\infty \exp\left(2 \int_0^t \frac{(e^{-u} - 1)}{u} du\right) dt.$$

A. Palásti [4] considered the corresponding problem in the two- and three-dimen-

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sional space and conjectured that the corresponding constants are  $\lambda^2$  and  $\lambda^3$ , respectively. Because of the occurrence of  $\lambda$  in these three cases and possibly in the  $n$ -dimensional case in general, it is desirable to evaluate  $\lambda$  as accurately as possible.

**2. Numerical Computation of  $M(x)$ .** For small values of  $x$ ,  $M(x)$  can be obtained by evaluating the integral in Eq. (1) recursively [5]. Since  $M(x) = 0$ ,  $0 \leq x < 1$ , we immediately obtain  $M(x)$ , for the interval  $1 \leq x < 2$ ,

$$M(x) = \frac{2}{x} \int_0^{x-1} M(t) dt + 1 = 1, \quad 1 \leq x < 2,$$

and

$$M(2) = 2 \left( \int_0^{1-\epsilon} M(t) dt + \int_{1-\epsilon}^{1+\epsilon} M(t) dt \right) + 1 = 1.$$

Thus,  $M(x) = 1$  for  $1 \leq x \leq 2$  and, using this in the integrand in Eq. (1), we obtain

$$M(x) = 3 - \frac{2}{(x-1)}, \quad 2 \leq x \leq 3,$$

$$M(x) = 7 - \frac{10 + 4\ln(x-2)}{(x-1)}, \quad 3 \leq x \leq 4.$$

For  $x > 4$ , the evaluation of successive integrals becomes tedious and a numerical method is most desirable. However, the above expressions are very useful for checking the accuracy of the numerical method to be employed.

For numerical integration of Eq. (1), we employ the recursion relation

$$(5) \quad \begin{aligned} (x+h)M(x+h+1) - xM(x+1) \\ = 2 \int_x^{x+h} M(t) dt + h \end{aligned}$$

and, putting  $x = 0, h, 2h, 3h, \dots$ ,  $M(x)$  was computed for  $x = (1+h)(h)50$ . The integral on the right side of Eq. (5) was approximated with a trapezoidal rule.

In order to improve the accuracy of  $M(x)$ , we perform four trapezoidal approximations to  $M(x)$  with  $h = 1/50, 1/100, 1/200$ , and  $1/400$  and use these results to eliminate terms of order  $h^2, h^4, h^6$ , in the Euler-Maclaurin integration formula. Thus, we obtain Romberg extrapolations of  $O(h^8)$  [6].

To check the over-all accuracy of the numerical integration of  $M(x)$ , it is essential to estimate the absolute error in Romberg extrapolation. In the present case, this is not possible since  $M(x)$  has discontinuities in the derivatives at  $x = 1, 2, 3, \dots$ , and we have to rely on some practical criterion. If the last term removed in the Euler-Maclaurin formula, i.e.,  $h^6$  with  $h = 1/400$  is used as the estimate of error in  $M(x)$ , we obtain an error  $\sim 2 \times 10^{-17}$  in  $M(x)$ . For  $x \leq 4$ , the numerical values of  $M(x)$  agreed with those obtained from the analytic expressions to 17D. All arithmetic was done with a word-length of 28 decimal digits in order to minimize the building up of round-off errors. Values of  $M(x)$  for  $x = 1(1)16$ , accurate to 17D, are given in Table 1.

TABLE 1  
 Values of  $M(x)$  and  $\epsilon(x)$   
 $M(x) = \bar{\lambda}(1 + x) - 1 + \epsilon(x)$ ;  $\bar{\lambda} = 0.74759\ 79202\ 53411\ 4351$

x	M(x)	$\epsilon(x)$
1	1.0000 0000 0000 0000 0	0.5048 0415 9493 1771 3
2	1.0000 0000 0000 0000 0	-0.2427 9376 0760 2343 1
3	1.9999 9999 9999 9999 9	0.0096 0831 8986 3542 5
4	2.7424 7042 5920 0729 2	0.0044 8082 4653 0157 4
5	3.4850 8592 4980 7502 6	-0.0005 0159 6539 7183 5
6	4.2331 8710 4299 1459 6	0.0000 0166 2525 2659 1
7	4.9807 8598 6275 6625 7	0.0000 0262 4248 3710 9
8	5.7283 8111 1921 1042 1	-0.0000 0017 0359 5987 1
9	6.4759 7920 4812 5400 1	0.0000 0000 2278 4256 6
10	7.2235 7712 3033 2178 9	0.0000 0000 0245 6921 0
11	7.9711 7504 3025 5138 7	-0.0000 0000 0015 4233 5
12	8.7187 7296 3294 6935 0	0.0000 0000 0000 3448 4
13	9.4663 7088 3547 7629 7	0.0000 0000 0000 0028 8
14	10.2139 6880 3801 1711 0	-0.0000 0000 0000 0004 3
15	10.9615 6672 4054 5829 7	0.0000 0000 0000 0000 1
16	11.7091 6464 4307 9943 9	-0.0000 0000 0000 0000 1

3. Evaluation of  $\lambda$ . Dvoretzky and Robbins [2] have shown that  $\lambda$  is given by

$$\text{Inf}_{z \leq t \leq z+1} \frac{M(t) + 1}{t + 1} \leq \lambda \leq \text{Sup}_{z \leq t \leq z+1} \frac{M(t) + 1}{(t + 1)}.$$

With the numerical values of  $M(t)$  for  $t = 2(.02)50$ , the ratios  $(M(t) + 1)/(t + 1)$  were computed. As  $t$  is increased, the bounds for  $\lambda$  are constantly narrowed. For  $t \geq 20$ , the first 19 decimal digits of the ratio  $(M(t) + 1)/(t + 1)$  remained stable. These stable digits define to 19D the accurate value  $\bar{\lambda}$ , of  $\lambda$ ,

$$\begin{aligned} \bar{\lambda} &= \int_0^\infty \exp\left(2 \int_0^t \frac{(e^{-u} - 1)}{u} du\right) dt \\ &= 0.74759, 79202, 53411, 4351. \end{aligned}$$

It is of interest to compute the error term  $\epsilon(x)$  which appears in the expression of  $M(x)$ ,

$$M(x) = \bar{\lambda}(1 + x) - 1 + \epsilon(x).$$

Values of  $\epsilon(x)$  for integral arguments are given in Table 1.

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**Added in proof.** Recent computations of Lal and Tiller give  $\bar{\lambda} = .74759, 79202, 53411, 43517, 87309, 43636, 52421, 02617, 2439$ , accurate to 44D.

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