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On Semicardinal Quadrature Formulae*

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Abstract. The present paper concerns the semicardinal quadrature formulae introduced in Part III of the reference [3]. These were the limiting forms of Sard's best quadrature formulae as the number of nodes increases indefinitely. Here we give a new derivation and characterization of these formulae. This derivation uses appropriate generating functions and also allows us to compute the coefficients very accurately.

Introduction. The present paper is a slightly shortened version of the MRC report [5]. Let m be a natural number and let

$$\mathcal{S}_{2m-1}^{+} = \{S(x)\} \tag{1}$$

denote the class of functions $S(x)$ satisfying the three conditions:

$$S(x) \in C^{2m-2}(\mathbf{R}), \tag{2}$$

$$S(x) \in \pi_{2m-1} \quad \text{in each of the intervals } (0, 1), (1, 2), \cdots, \tag{3}$$

$$S(x) \in \pi_{m-1} \quad \text{in the interval } (-\infty, 0). \tag{4}$$

These functions are the so-called *natural* semicardinal splines of degree $2m - 1$.

It was shown in [3, Lemma 5, Section 9] that if

$$S(x) \in \mathcal{S}_{2m-1}^{+} \cap L_1(\mathbf{R}^+), \tag{5}$$

then

$$\sum_{\nu=0}^{\infty} |S(\nu)| < \infty. \tag{6}$$

It follows that, if B_ν is a sequence of constants such that

$$B_\nu = O(1) \quad \text{as } \nu \to \infty, \tag{7}$$

then the functional

$$RS = \int_0^\infty S(x)\,dx - \sum_0^\infty B_\nu S(\nu) \tag{8}$$

is well defined for every $S(x)$ satisfying (5).

In the same paper [3, Theorem 6, Section 10], the following theorem was established.

THEOREM 1. *We consider a quadrature formula*

$$\int_0^\infty f(x)\,dx = \sum_0^\infty B_\nu f(\nu) + Rf \tag{9}$$

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with perfectly arbitrary constant coefficients B_ν *subject only to the condition* (7). *Among these formulae, there is exactly one with the property that*

(10) $$Rf = 0 \quad \textit{whenever } f(x) \in \mathcal{S}^{+}_{2m-1} \cap L_1(\mathbf{R}^{+}).$$

We denote this unique formula by

(11) $$\int_0^\infty f(x)\,dx = \sum_0^\infty H_\nu^{(m)} f(\nu) + Rf$$

and call it the semicardinal quadrature formula of order *m*.

For the derivation of (11) by integra ing the semicardinal *interpolation* formula, see [3, Section 10], wherein its connection with some conjectures due to L. F. Meyers and A. Sard concerning *best* quadrature formulae is explained (see also [4, Lecture 8]). The purpose of the present note is the accurate computation of the values of the coefficients $H_\nu^{(m)}$ for $m = 2, 3, \cdots, 7$. The tables of Sections 7 and 8 are based on computations beautifully performed by Mrs. Julia Gray, of the Computing Staff of the Mathematics Research Center, on the CDC 3600. They were done in double precision and all decimals listed should be correct, as we had anywhere from 17 to 24 correct decimals throughout. The zeros of the Euler-Frobenius polynomials of Section 7 were found by the algorithm due to D. H. Lehmer. It seems of some interest to observe that

$$H_4^{(7)} < 0.$$

We also give a new proof of Theorem 1 which is simpler than the proof presented in [3, Section 10] where the main emphasis was in establishing the harder Meyers-Sard conjectures.

We conclude this Introduction by mentioning two further remarkable semicardinal formulae: The first is the *Euler-Maclaurin* formula

(12) $$\int_0^\infty f(x)\,dx = \tfrac{1}{2}f(0) + f(1) + f(2) + \cdots + \sum_{r=1}^{m-1} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(0) + Rf.$$

The second is the so-called *complete* semicardinal formula

(13) $$\int_0^\infty f(x)\,dx = \sum_0^\infty \tilde{H}_\nu^{(m)} f(\nu) + \sum_{i=1}^{m-1} A_i^{(m)} f^{(i)}(0) + Rf.$$

Both formulae are uniquely defined among quadrature formulae of their type (i.e., when all their terms are provided with arbitrary coefficients subject only to the condition that the coefficients of $f(\nu)$ should form a bounded sequence) by the condition of being exact, hence $Rf = 0$, whenever $f(x)$ is *any* spline of degree $2m - 1$ in the interval $[0, +\infty)$, with knots at $1, 2, \cdots$, such that $f(x) \in L_1(\mathbf{R}^+)$. Among the formulae (11), (12), and (13), the formula (13) is, as a rule, the most accurate in numerical applications (after an appropriate change of step), while (11) is the least accurate. The computation of the coefficients of the complete formula (13) is the subject of Silliman's forthcoming paper [6].

The reader will see that the use of the *B*-splines (Section 1) transforms a fairly formidable problem into one that is within easy reach of the Euler-Laplace method of generating functions.

I. The Construction of the Semicardinal Quadrature Formula.

1. *B-Splines and Euler-Frobenius Polynomials.* Here we collect tools and results that have proved to be indispensable in the study of cardinal splines. Writing $x_+ = \max(0, x)$, the forward *B-spline* is defined by (see [1, Section 1])

$$Q_m(x) = \frac{1}{(m-1)!} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (x-i)_+^{m-1} \qquad (x \in \mathbf{R}). \tag{1.1}$$

This is a spline function of degree $m-1$, with knots at $x = 0, 1, \cdots, m$. The symmetry property $Q_m(x) = Q_m(m-x)$ shows that we may equivalently write it in the form

$$Q_n(x) = \frac{1}{(m-1)!} \sum_{0}^{m} (-1)^{m-\nu} \binom{m}{\nu} (\nu - x)_+^{m-1}. \tag{1.2}$$

This is a frequency function. More precisely,

$$Q_m(x) > 0 \quad \text{if } 0 < x < m, \qquad Q_m(x) = 0 \quad \text{if } x \leqq 0, \text{ or } x \geqq m. \tag{1.3}$$

Euler's generating function

$$\frac{x-1}{x-e^z} = \sum_{0}^{\infty} \frac{\Pi_n(x)}{(x-1)^n} \frac{z^n}{n!} \tag{1.4}$$

defines the polynomial $\Pi_n(x)$ of degree $n-1$, called the *Euler-Frobenius polynomial.* For proofs of its properties described below in Lemma 1, we refer to [2, Lemma 7].

LEMMA 1. (i) *$\Pi_n(x)$ is a reciprocal monic polynomial of degree $n-1$ with integer coefficients satisfying the recurrence relation*

$$\Pi_{n+1}(x) = (1+nx)\Pi_n(x) + x(1-x)\Pi_n'(x) \qquad (\Pi_1(x) = 1). \tag{1.5}$$

(ii) *The identity*

$$\Pi_n(x)/(1-x)^{n+1} = \sum_{0}^{\infty} (\nu+1)^n x^\nu \qquad (|x| < 1), \tag{1.6}$$

holds.

(iii) *The zeros λ_ν of $\Pi_n(x)$ are all simple and negative. We label them so that*

$$\lambda_{n-1} < \lambda_{n-2} < \cdots < \lambda_2 < \lambda_1 < 0. \tag{1.7}$$

(iv) *The identity*

$$\Pi_n(x) = n! \sum_{0}^{n-1} Q_{n+1}(\nu+1)x^\nu \tag{1.8}$$

holds.

The identity (1.8) shows the close relation between B-splines and Euler-Frobenius polynomials. In Section 7, the reader will find the polynomials $\Pi_{2m-1}(x)$ and their zeros for $m = 2, 3, 4, 5, 6,$ and 7.

2. *A Recurrence Relation.* In Sections 2, 3, and 4, we determine the Q.F. (9) satisfying conditions (7) and (10). To begin with, we ignore condition (7) and argue as follows.

We integrate the B-spline (1.2) m times so as to preserve the vanishing of the function in $(m, +\infty)$. This condition uniquely defines the integral

$$\sigma(x) = \sigma_m(x) = \frac{1}{(2m-1)!} \sum_0^m (-1)^\nu \binom{m}{\nu} (\nu - x)_+^{2m-1} \tag{2.1}$$

having the properties

$$\sigma^{(m)}(x) = Q_m(x), \qquad \sigma(x) = 0 \quad \text{if } x \geqq m. \tag{2.2}$$

Moreover, since $Q_m(x) = 0$ if $x \leqq 0$, we conclude that

$$\sigma(x) \in \mathcal{S}_{2m-1}^+ \cap L_1(\mathbf{R}^+). \tag{2.3}$$

Clearly, this property of $\sigma(x)$ will remain valid if we shift its graph *to the right* by an integer amount, hence

$$\sigma(x-n) \in \mathcal{S}_{2m-1}^+ \cap L_1(\mathbf{R}^+) \quad \text{for } n = 0, 1, 2, \cdots. \tag{2.4}$$

We conclude: *The coefficients B_ν of an arbitrary Q.F.* (9), (7), *that enjoys the property* (10), *must satisfy the relations*

$$\int_0^{n+m} \sigma(x-n)\,dx = \sum_{\nu=0}^{n+m-1} B_\nu \sigma(\nu - n) \qquad (n = 0, 1, 2, \cdots). \tag{2.5}$$

The series on the right side indeed breaks off as indicated because of the second relation (2.2).

3. *The Summation of Certain Power Series.* The structure of the relations (2.5) suggests the use of generating functions for the determination of the B_ν. Indeed, the right side of (2.5) is seen to be equal to the coefficient of x^{n+m-1} in the product of power series

$$\left(\sum_0^\infty B_i x^i\right)\left(\sum_0^\infty \sigma(m-1-\nu)x^\nu\right). \tag{3.1}$$

(A) To simplify notations, we define the sequence $(s_\nu;\ \nu = 0, 1, \cdots)$ by

$$s_\nu = (-1)^m(2m-1)!\,\sigma(m-1-\nu) = \sum_{r=0}^m (-1)^{m+r}\binom{m}{r}(r - m + 1 + \nu)_+^{2m-1}, \tag{3.2}$$

or

$$s_\nu = \sum_{k=0}^m (-1)^k \binom{m}{k} (\nu + 1 - k)_+^{2m-1}. \tag{3.3}$$

Multiplying by x^ν and summing for $\nu = 0, 1, \cdots$, we obtain

$$\begin{aligned} \sum_0^\infty s_\nu x^\nu &= \sum_{k=0}^m (-1)^k \binom{m}{k} \sum_{\nu=0}^\infty (\nu + 1 - k)_+^{2m-1} x^\nu \\ &= \sum_{k=0}^m (-1)^k \binom{m}{k} \sum_{\nu=0}^\infty (\nu+1)^{2m-1} x^{\nu+k}. \end{aligned}$$

Using (1.6), the right side becomes

$$= \sum_{k=0}^m (-1)^k \binom{m}{k} x^k \Pi_{2m-1}(x)/(1-x)^{2m}$$

and we finally obtain that

$$\sum_0^\infty s_\nu x^\nu = \Pi_{2m-1}(x)/(1-x)^m. \tag{3.4}$$

(B) For the integrand on the left side of (2.5), we find, by (2.1),

$$\sigma(x-n) = \frac{1}{(2m-1)!}\sum_{\nu=0}^m (-1)^\nu \binom{m}{\nu}(\nu+n-x)_+^{2m-1},$$

whence

$$\int_0^{n+m} \sigma(x-n)\,dx = \frac{1}{(2m)!}\sum_{\nu=0}^m (-1)^\nu\binom{m}{\nu}(\nu+n)_+^{2m}. \tag{3.5}$$

As in (3.2), we introduce the new quantities

$$\begin{aligned} F_{n+m-1} &= (-1)^m(2m-1)!\int_0^{n+m}\sigma(x-n)\,dx \\ &= \frac{1}{2m}\sum_{\nu=0}^m (-1)^{m+\nu}\binom{m}{\nu}(\nu+n)_+^{2m} \qquad (n \geqq 0), \end{aligned} \tag{3.6}$$

and wish to sum the series

$$\sum_{n=0}^\infty F_{n+m-1}x^{n+m-1}. \tag{3.7}$$

From (3.6) we obtain

$$2m\sum_{n=0}^\infty F_{n+m-1}x^{n+m-1} = \sum_{\nu=0}^m (-1)^{m+\nu}\binom{m}{\nu}\sum_{n=0}^\infty (\nu+n)_+^{2m}x^{n+m-1}, \tag{3.8}$$

while the inside sum is

$$\begin{aligned} \sum_{n=0}^\infty (\nu+n)_+^{2m}x^{n+m-1} &= x^{m-\nu}\sum_{n=0}^\infty (n+\nu)_+^{2m}x^{n+\nu-1} \\ &= x^{m-\nu}\sum_{r=0}^\infty (r+1)^{2m}x^r - x^{m-\nu}\sum_{r=0}^{\nu-2}(r+1)^{2m}x^r \\ &= x^{m-\nu}\Pi_{2m}(x)/(1-x)^{2m+1} - V_\nu(x) \end{aligned}$$

by (1.6). Here, $V_\nu(x)$ is an element of π_{m-2}. Substituting this into (3.8), we obtain

$$\sum_{m-1}^\infty F_\nu x^\nu = \frac{1}{2m}\frac{\Pi_{2m}(x)}{(1-x)^{m+1}} - \frac{1}{2m}V(x), \quad \text{where } V(x) \in \pi_{m-2}. \tag{3.9}$$

Evidently, $V(x)$ is such as to cancel the first $m-1$ terms of the power series expansion of the first term on the right side.

The relations (2.5) may now be written as

$$F_n = \sum_{\nu=0}^n B_\nu s_{n-\nu} \quad \text{for } n \geqq m-1. \tag{3.10}$$

We may here select the first $m-1$ terms

$$B_0,\ B_1,\ \cdots,\ B_{m-2} \tag{3.11}$$

arbitrarily and determine the entire sequence (B_ν) *recursively by* (3.10). Equivalently, we may select the $m-1$ quantities $F_0, F_1, \cdots, F_{m-2}$, arbitrarily and determine (B_ν) from the identity

$$\sum_0^\infty F_\nu x^\nu = \left(\sum_0^\infty B_\nu x^\nu\right)\left(\sum_0^\infty s_\nu x^\nu\right). \tag{3.12}$$

By (3.4) and (3.9), we have

$$\sum_0^\infty s_\nu x^\nu = \Pi_{2m-1}(x)/(1-x)^m, \tag{3.13}$$

and

$$\sum_0^\infty F_\nu x^\nu = \frac{1}{2m}\frac{\Pi_{2m}(x)}{(1-x)^{m+1}} - \frac{1}{2m}U(x), \tag{3.14}$$

where U is an arbitrary element of π_{m-2}. Solving (3.12) for $\sum_0^\infty B_\nu x^\nu$, we obtain the following:

THEOREM 2. *The coefficients* (B_ν) *of the most general functional*

$$Rf = \int_0^\infty f(x)\,dx - \sum_0^\infty B_\nu f(\nu), \tag{3.15}$$

that vanishes for the functions of the sequence

$$\sigma(x-n) \qquad (n = 0, 1, 2, \cdots), \tag{3.16}$$

are the expansion coefficients of

$$R_m(x) = \sum_0^\infty B_\nu x^\nu \tag{3.17}$$

where

$$R_m(x) = \frac{\Pi_{2m}(x)}{2m(1-x)\Pi_{2m-1}(x)} - \frac{(1-x)^m U(x)}{2m\Pi_{2m-1}(x)}. \tag{3.18}$$

Here, $U(x)$ *is an arbitrary element of* π_{m-2}.

4. *Determining the Coefficients* $H_\nu^{(m)}$. This will be done by requiring the coefficients (B_ν) of (3.17) to satisfy (7) or

$$B_\nu = O(1) \quad \text{as } \nu \to \infty. \tag{4.1}$$

The order of magnitude of the B_ν for large ν is controlled by the location of the poles of the rational function $R_m(x)$. Let us first transform its expression slightly. From the recurrence relation (1.5), we find that

$$\Pi_{2m}(x) = (1 + (2m-1)x)\Pi_{2m-1}(x) + x(1-x)\Pi'_{2m-1}(x),$$

and, substituting into (3.18), we obtain that

$$R_m(x) = \frac{1+(2m-1)x}{2m(1-x)} + \frac{x\Pi'_{2m-1}(x)}{2m\Pi_{2m-1}(x)} - \frac{(1-x)^m}{2m\Pi_{2m-1}(x)}U(x). \tag{4.2}$$

From (1.7) we know that the $2m-2$ zeros λ_ν of $\Pi_{2m-1}(x)$ are simple and negative. Also, that $\Pi_{2m-1}(x)$ is a *reciprocal* polynomial, whence the relations $\lambda_1\lambda_{2m-2} =$

$\lambda_2\lambda_{2m-3} = \cdots = \lambda_{m-1}\lambda_m = 1$. It follows that these zeros satisfy the inequalities

$$\lambda_{2m-2} < \cdots < \lambda_m < -1 < \lambda_{m-1} < \cdots < \lambda_1 < 0. \tag{4.3}$$

Let

$$U(x) = a_0 + a_1 x + \cdots + a_{m-2}x^{m-2}. \tag{4.4}$$

It is now easy to decompose $R_m(x)$ into partial fractions. Observing that $R_m(x)$ is regular at $x = \infty$, we find that

$$\begin{aligned} R_m(x) = & -\frac{1}{2m} + \frac{1}{2m}(-1)^{m+1}a_{m-2} + \frac{1}{1-x} \\ & + \frac{1}{2m}\sum_1^{2m-2}\frac{\lambda_\nu}{x-\lambda_\nu} - \frac{1}{2m}\sum_1^{2m-2}\frac{U(\lambda_\nu)(1-\lambda_\nu)^m}{(x-\lambda_\nu)\Pi'_{2m-1}(\lambda_\nu)}. \end{aligned} \tag{4.5}$$

From (4.3) we see that the poles $\lambda_1, \cdots, \lambda_{m-1}$ are inside the unit circle, while $\lambda_m, \cdots, \lambda_{2m-2}$ are outside. Also, $x = 1$ is a simple pole, by (4.5). It follows that (4.1) will hold if and only if the polynomial $U(x)$ can be so chosen that the inside poles $\lambda_1, \cdots, \lambda_{m-1}$ cancel out, i.e., their residues vanish. An inspection of (4.5) shows this to be the case if and only if $U(x)$ satisfies the equations

$$U(\lambda_\nu) = \lambda_\nu \Pi'_{2m-1}(\lambda_\nu)(1-\lambda_\nu)^{-m} \qquad (\nu = 1, \cdots, m-1). \tag{4.6}$$

We see that $U(x)$ exists uniquely, because (4.6) describes an ordinary Lagrange interpolation problem. This establishes

THEOREM 3. *There is a unique Q.F.*

$$\int_0^\infty f(x)\,dx = \sum_0^\infty H_\nu^{(m)} f(\nu) + Rf \tag{4.7}$$

having bounded coefficients and which is exact for the sequence of functions $\sigma(x-n)$ $(n = 0, 1, \cdots)$. *Its coefficients are given by the expansion*

$$R_m(x) = \sum_0^\infty H_\nu^{(m)} x^\nu \qquad (|x| < 1). \tag{4.8}$$

Here

$$\begin{aligned} R_m(x) = & -\frac{1}{2m} + \frac{1}{2m}(-1)^{m+1}a_{m-2} + \frac{1}{1-x} \\ & + \frac{1}{2m}\sum_{\nu=m}^{2m-2}\left\{\lambda_\nu - \frac{U(\lambda_\nu)(1-\lambda_\nu)^m}{\Pi'_{2m-1}(\lambda_\nu)}\right\}\frac{1}{x-\lambda_\nu}, \end{aligned} \tag{4.9}$$

where $U(x) = a_{m-2}x^{m-2}$ + *lower degree terms, is the solution of the interpolation problem* (4.6).

In order to complete a proof of Theorem 1, we are still to show that the remainder functional Rf of the formula (4.7) satisfies the condition (10) of Theorem 1. For a proof of this, we refer to [5, Section 5].

5. *Final Computational Details.* We return to the rational function $R_m(x)$, defined by (4.9), that generates the $H_\nu^{(m)}$ by (4.8). For even moderately large values of m, the zero λ_1 is small and its reciprocal λ_{2m-2} is correspondingly large (e.g., for $m = 7$, we find that $\lambda_1 = -.0001251$). It is therefore important from the computational

point of view to express the right side of (4.9) in terms of the zeros $\lambda_1, \cdots, \lambda_{m-1}$. This is easily done by the following device: We define the new polynomials U^* and $\Pi_{2m-1}{}^*$ by setting

$$U^*(x) = x^{m-2}U(x^{-1}), \qquad \Pi^*_{2m-1}(x) = x^{2m-3}\Pi'_{2m-1}(x^{-1}). \tag{5.1}$$

In terms of these polynomials, (4.9) becomes

$$R_m(x) = C + \frac{1}{1-x} + \sum_{\nu=1}^{m-1} C_\nu \frac{1}{1-\lambda_\nu x}, \tag{5.2}$$

where

$$C = -\frac{1}{2m} + \frac{1}{2m}(-1)^{m+1}a_{m-2}, \tag{5.3}$$

$$C_\nu = \frac{1}{2m}\left\{\frac{U^*(\lambda_\nu)(\lambda_\nu - 1)^m}{\Pi^*_{2m-1}(\lambda_\nu)} - 1\right\} \qquad (\nu = 1, \cdots, m-1). \tag{5.4}$$

Expanding the right side of (5.2) in powers of x and using (4.8), we obtain

COROLLARY 1. *The coefficients of the semicardinal Q.F.* (11) *have the values*

$$H_0^{(m)} = C + 1 + \sum_{\nu=1}^{m-1} C_\nu, \tag{5.5}$$

$$H_j^{(m)} = 1 + \sum_{\nu=1}^{m-1} C_\nu\lambda_\nu^j \qquad (j = 1, 2, \cdots), \tag{5.6}$$

where C and C_ν are given by (5.3), (5.4).

It is convenient to define

$$h_0^{(m)} = H_0^{(m)} - \tfrac{1}{2}, \qquad h_j^{(m)} = H_j^{(m)} - 1 \qquad (j \geq 1), \tag{5.7}$$

and to write the Q.F. (11) in the form

$$\int_0^\infty f(x)\,dx = T + \sum_{\nu=0}^{\infty} h_\nu^{(m)} f(\nu) + Rf, \tag{5.8}$$

where T stands for the trapezoidal sum

$$T = \tfrac{1}{2}f(0) + \sum_1^\infty f(\nu). \tag{5.9}$$

From (5.5), (5.6) and in view of (5.7), we obtain that

$$h_0^{(m)} = C + \tfrac{1}{2} + \sum_{\nu=1}^{m-1} C_\nu, \tag{5.10}$$

$$h_j^{(m)} = \sum_{\nu=1}^{m-1} C_\nu\lambda_\nu^j \qquad (j = 1, 2, \cdots). \tag{5.11}$$

6. *The Case $m = 2$ of Cubic Splines.* We mention this case separately because the results are explicit and also because, for this case, Meyers and Sard established their conjecture. From our formulae (4.4), (4.6), (5.3) to (5.6), we easily find that

$$\lambda_1 = -2 + \sqrt{3}, \qquad a_0 = -\tfrac{1}{3}\sqrt{3}, \qquad C = \frac{-3 + \sqrt{3}}{12}, \qquad C_1 = -\tfrac{1}{2},$$

and therefore

$$H_0^{(2)} = \frac{3 + \sqrt{3}}{12}, \qquad H_j^{(2)} = 1 - \tfrac{1}{2}\lambda_1^j \qquad (j = 1, 2, \cdots).$$

These agree with the values given by Meyers and Sard. For references to the work of Meyers and Sard, see [3].

II. Numerical Results.

7. *The Polynomials* $\Pi_{2m-1}(x)$ *and Their Zeros for* $m = 2, 3, \cdots, 7$.

$m = 2$: $\Pi_3(x) = x^2 + 4x + 1$

ν	λ_ν			
1	-.26794	91924	31123	
2	-3.73205	08075	68877	

$m = 3$: $\Pi_5(x) = x^4 + 26x^3 + 66x^2 + 26x + 1$

ν	λ_ν			
1	-.04309	62882	03264	7
2	-.43057	53470	99974	
3	-2.32247	38869	40428	
4	-23.20385	44777	56334	

$m = 4$: $\Pi_7(x) = x^6 + 120x^5 + 1191x^4 + 2416x^3 + 1191x^2 + 120x + 1$

ν	λ_ν			
1	-.00914	86948	09608	28
2	-.12255	46151	92326	69
3	-.53528	04307	96438	17
4	-1.86817	96353	21453	
5	-8.15962	74316	61271	
6	-109.30520	91922	18903	

$m = 5$: $\Pi_9(x) = \sum_0^8 c_\nu x^\nu$

$$1 = c_0 = c_8$$
$$502 = c_1 = c_7$$
$$14\ 608 = c_2 = c_6$$
$$88\ 234 = c_3 = c_5$$
$$156\ 190 = c_4$$

ν	λ_ν			
1	-.00212	13069	03180	8184
2	-.04322	26085	40481	7521
3	-.20175	05201	93153	2388
4	-.60799	73891	68625	78
5	-1.64474	39048	50311	
6	-4.95661	67117	81528	
7	-23.13603	99977	57483	
8	-471.40750	75608	05236	

$m = 6$: $\Pi_{11}(x) = \sum_0^{10} c_\nu x^\nu$

$$1 = c_0 = c_{10}$$
$$2\ 036 = c_1 = c_9$$
$$152\ 637 = c_2 = c_8$$
$$2\ 203\ 488 = c_3 = c_7$$
$$9\ 738\ 114 = c_4 = c_6$$
$$15\ 724\ 248 = c_5$$

ν	λ_ν			
1	-.00051	05575	34446	50206
2	-.01666	96273	66234	65610
3	-.08975	95997	93713	30994
4	-.27218	03492	94785	88569
5	-.66126	60689	00734	70691
6	-1.51225	05857	02007	
7	-3.67403	45237	66984	
8	-11.14086	96373	22505	
9	-59.98934	33746	19208	
10	-1958.64311	56756	99381	

$m = 7$: $\Pi_{13}(x) = \sum_{0}^{12} c_\nu x^\nu$

$1 = c_0 = c_{12}$

$8\ 178 = c_1 = c_{11}$

$1\ 479\ 726 = c_2 = c_{10}$

$45\ 533\ 450 = c_3 = c_9$

$423\ 281\ 535 = c_4 = c_8$

$1\ 505\ 621\ 508 = c_5 = c_7$

$2\ 275\ 172\ 004 = c_6$

ν	λ_ν				
1	-.00012	51001	13214	41871	596
2	-.00673	80314	15244	91399	848
3	-.04321	38667	40363	66964	776
4	-.13890	11131	94319	43021	
5	-.33310	72329	30623	59248	
6	-.70189	42518	16807	86245	
7	-1.42471	60414	99933		
8	-3.00203	62848	38854		
9	-7.19936	63477	77381		
10	-23.14072	02231	67524		
11	-148.41129	97362	23031		
12	-7993.59788	17702	82704		

8. *The Numerical Values of* $h_0^{(m)} = H_0^{(m)} - \frac{1}{2}$, $h_j^{(m)} = H_j^{(m)} - 1$ $(j \geq 1)$, *for* $m = 2, 3, \cdots, 7$. We have written the Q.F. (11) in the form (5.8), (5.9), where the coefficients $h_\nu^{(m)}$ are defined by (5.7). Below we give the values of the coefficients $C, C_1, \cdots, C_{m-1}$, appearing in the formulae (5.10), (5.11), which were used throughout the computation. The corresponding λ_ν, for each m, are known from Section 7.

m = 2: C = -.10566 24327 02594

C_1 = -.50000 00000 00000

j	$10^9 \cdot h_j^{(2)}$	j	$10^9 \cdot h_j^{(2)}$	j	$10^9 \cdot h_j^{(2)}$	j	$10^9 \cdot h_j^{(2)}$
0	-105 662 433	4	-2 577 388	8	-13 286	12	-68
1	133 974 596	5	690 609	9	3 560	13	18
2	-35 898 385	6	-185 048	10	-954	14	-5
3	9 618 943	7	49 583	11	256	15	1

m = 3: C = -1.55683 40723 44085

C_1 = 1.61253 86058 42966

C_2 = -.69966 76766 67689

j	$10^9 \cdot h_j^{(3)}$	j	$10^9 \cdot h_j^{(3)}$	j	$10^9 \cdot h_j^{(3)}$	j	$10^9 \cdot h_j^{(3)}$
0	-143 963 143	7	1 919 711	14	-5 267	21	14
1	231 765 224	8	-826 580	15	2 268	22	-6
2	-126 720 028	9	355 905	16	-977	23	3
3	55 723 001	10	-153 244	17	420	24	-1
4	-24 042 963	11	65 983	18	-181		
5	10 354 462	12	-28 411	19	78		
6	-4 458 469	13	12 233	20	-34		

m = 4: C = 29.79116 16580 89087

C_1 = -34.33080 08334 22275

C_2 = 5.03831 17952 59740

C_3 = -1.16658 41207 39341

j	$10^9 \cdot h_j^{(4)}$	j	$10^9 \cdot h_j^{(4)}$	j	$10^9 \cdot h_j^{(4)}$	j	$10^9 \cdot h_j^{(4)}$
0	-167 911 501	9	4 208 672	18	-15 184	27	54
1	321 063 307	10	-2 252 833	19	8 128	28	-29
2	-261 455 521	11	1 205 899	20	-4 351	29	16
3	169 672 663	12	-645 494	21	2 329	30	-8
4	-94 636 306	13	345 520	22	-1 247	31	4
5	51 125 936	14	-184 950	23	667	32	-2
6	-27 424 202	15	99 000	24	-357	33	1
7	14 686 684	16	-52 993	25	191		
8	-7 862 358	17	28 366	26	-102		

$m = 5$: $C = -1185.60066\ 69187\ 87416$

$C_1 = 1278.39413\ 47945\ 01574$

$C_2 = -104.82413\ 90602\ 21726$

$C_3 = 13.49945\ 93367\ 63573$

$C_4 = -2.15378\ 46634\ 43702$

j	$10^9 \cdot h_j^{(5)}$	j	$10^9 \cdot h_j^{(5)}$	j	$10^9 \cdot h_j^{(5)}$	j	$10^9 \cdot h_j^{(5)}$
0	-184 996 511	11	9 038 711	22	-37 935	33	159
1	404 878 934	12	-5 495 637	23	23 064	34	-97
2	-436 776 761	13	3 341 358	24	-14 023	35	59
3	381 665 032	14	-2 031 542	25	8 526	36	-36
4	-272 313 302	15	1 235 173	26	-5 184	37	22
5	174 445 008	16	-750 982	27	3 152	38	-13
6	-107 886 251	17	456 595	28	-1 916	39	8
7	65 963 996	18	-277 609	29	1 165	40	-5
8	-40 180 533	19	168 785	30	-708	41	3
9	24 444 711	20	-102 621	31	431	42	-2
10	-14 865 358	21	62 393	32	-262	43	1

m = 6: C = 75691.58329 09095 55732

C_1 = -78988.38815 48082 40699

C_2 = 3556.66826 01136 24533

C_3 = -291.24484 63712 04503

C_4 = 34.93429 00662 62594

C_5 = -4.25089 68338 22148

j	$10^9 \cdot h_j^{(6)}$	j	$10^9 \cdot h_j^{(6)}$	j	$10^9 \cdot h_j^{(6)}$	j	$10^9 \cdot h_j^{(6)}$
0	-198 056 924	14	-12 993 723	28	-39 721	42	-121
1	484 349 563	15	8 592 475	29	26 266	43	80
2	-649 567 273	16	-5 681 957	30	-17 369	44	-53
3	718 914 116	17	3 757 298	31	11 485	45	35
4	-639 708 909	18	-2 484 577	32	-7 595	46	-23
5	486 987 860	19	1 642 967	33	5 022	47	15
6	-341 365 669	20	-1 086 439	34	-3 321	48	-10
7	231 172 876	21	718 425	35	2 196	49	7
8	-154 363 128	22	-475 070	36	-1 452	50	-4
9	102 483 801	23	314 148	37	960	51	3
10	-67 880 429	24	-207 735	38	-635	52	-2
11	44 917 348	25	137 368	39	420	53	1
12	-29 710 573	26	-90 837	40	-278	54	-1
13	19 648 841	27	60 067	41	184	55	1

m = 7: C = -71 24756.13044 78377 42764

C_1 = 72 97768.36410 88111 56638

C_2 = -1 81492.08505 99971 63019

C_3 = 9205.14045 15342 90528

C_4 = -806.75362 55949 48760

C_5 = 89.55157 21836 35045

C_6 = -8.79549 99208 94050

j	$10^9 \cdot h_j^{(7)}$	j	$10^9 \cdot h_j^{(7)}$	j	$10^9 \cdot h_j^{(7)}$	j	$10^9 \cdot h_j^{(7)}$
0	-208 500 822	17	21 422 260	34	-52 179	51	127
1	560 220 481	18	-15 036 414	35	36 624	52	-89
2	-897 279 922	19	10 554 057	36	-25 706	53	63
3	1 206 104 998	20	-7 407 860	37	18 043	54	-44
4	-1 300 751 517	21	5 199 544	38	-12 664	55	31
5	1 171 420 907	22	-3 649 533	39	8 889	56	-22
6	-935 088 480	23	2 561 587	40	-6 239	57	15
7	698 229 096	24	-1 797 964	41	4 379	58	-11
8	-504 660 854	25	1 261 980	42	-3 074	59	7
9	359 162 044	26	-885 777	43	2 157	60	-5
10	-253 752 687	27	621 722	44	-1 514	61	4
11	178 661 845	28	-436 383	45	1 063	62	-3
12	-125 586 597	29	306 295	46	-746	63	2
13	88 210 126	30	-214 986	47	524	64	-1
14	-61 934 710	31	150 898	48	-368	65	1
15	43 478 456	32	-105 914	49	258	66	-1
16	-30 519 557	33	74 341	50	-181		

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